# Quantum Super-Resolution Imaging of Two-Incoherent Point-Sources

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Traditional imaging systems are resolution-limited by the Rayliegh Criterion which defines the minimum separation distance at which two distant independently radiating point sources can still be resolved. In this lab we will formulate the two point-source imaging problem quantum mechanically and discover that the Rayleigh criterion does *not* constitute a fundamental physical limit to resolution. Instead, we will come to find that the information contained in any detected photon about the separation between the point sources actually remains constant for all separation distances. Finally, we will explore how a certain quantum measurement scheme called spatial mode demultiplexing (SPADE) allows us to saturate these information-theoretic limits.

**Key Terms:** quantum estimation, Fisher information, Rayleigh limit

## I. INTRODUCTION

Historically, the task of imaging two point sources has been used to derive and measure the resolution limits of direct imaging systems. In this context, direct imaging systems have technical meaning - they are the class of optical systems that focus light from the object plane onto a detector in a 1-to-1 fashion so as to form an image. In the quantum imaging literature, this is also referred to as Direct Detection. Due to Fraunhofer diffraction effects, direct imaging systems with finite apertures have a Point Spread Function (PSF) with a characteristic non-zero width. Therefore, direct imaging never truly achieves a 1-to-1 mapping of the object plane onto the detector. Instead, the blur generated by the PSF defines a resolution limit described by the Rayleigh criterion.

The canonical two-point-source problem is setup as follows. Suppose we wish to image two distant incoherent sources $s_1$ and $s_2$. To idealize the problem further, we make a few simplifying assumptions.

- The sources have equal brightness - that is, the mean number of photons emitted by each source per unit time is the same.
- The rotational degree of freedom is fixed - that is we know which axis in the object plane the two sources are separated along. This allows us to formulate the problem in 1D instead of over the entire 2D image plane.
- The midpoint between the two sources is known exactly and coincides with the optical axis of the imaging system. So the only degree of freedom is the source separation.

Under these assumptions, the task is to estimate the angular separation $\theta$ between the two sources. For the remainder of this exploration, we will assume that the PSF of the imaging system we are using can, to a good approximation, be modelled with a gaussian PSF.

$$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-(\frac{x}{2\sigma})^2}$$

Here $x$ is a position coordinate (units of radians) on the image plane and $\sigma$ is the standard deviation of the PSF (units of radians). The Rayleigh criterion for such a system says that the minimum resolvable angular separation $\theta_R$ is given by

$$\theta_R \approx \sigma$$

If the system is shift-invariant, then $s_1$ and $s_2$ register a normalized intensity pattern over the image axis.

$$\nu(x;\theta) = \frac{1}{2}\left|\psi\left(x - \frac{\theta}{2}\right)\right|^2 + \frac{1}{2}\left|\psi\left(x + \frac{\theta}{2}\right)\right|^2 \quad (1)$$

The prefactors of 1/2 come from the sources having equal brightness. We can understand the Rayleigh criterion visually as the separation at which there are no longer two distinguishable peaks in $\nu(x;\theta)$ as shown in figure 1

Note that $\nu(x;\theta)$ effectively forms a probability density over the image axis for where we may detect a photon. There is a direct link between this probability distribution and the quantum mechanical description of the field in the position representation which we will explore shortly.

## II. THEORY

### A. Formulating a Quantum Field

Light is fundamentally a quantum phenomenon. As such, a fully accurate description of the field at the image plane involves assigning it a quantum state. In particular, we will describe the field as a mixed state using

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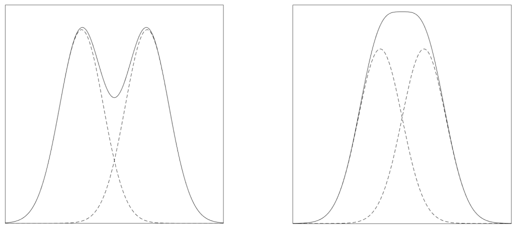

**FIG. 1:** Intensity distributions at the image plane for two sources. On the left the sources are separated by more then the Rayleigh Limit and we see two distinguishable peaks (solid line) in the intensity distribution. On the right, the sources are separated by exactly the Rayleigh limit $\theta = \theta_R = \sigma$ at which point we can no longer discern two peaks in the intensity distribution.

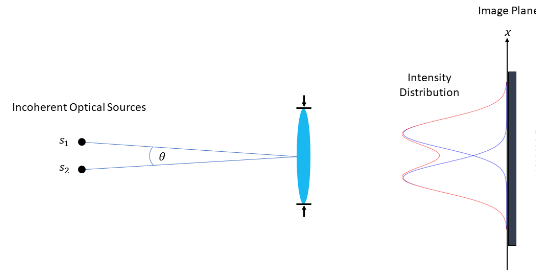


**FIG. 2:** A diagram of a direct imaging setup. Photons are registered spatially on a detector.

the density operator formalism. Consider observing the field over a short temporal interval so that at any point within the interval the field is either in the vacuum state $\hat{\rho}_0 = |\text{vac}\rangle\langle\text{vac}|$ with probability $1-\epsilon$ or in a single-photon excited state $\hat{\rho}_1$ with probability $\epsilon << 1$. This short temporal interval ensures that with overwhelming probability we never find the field populated with more than one photon. To a good approximation, the field can then be described with the density operator,

$$\hat{\rho} \approx (1-\epsilon)\hat{\rho}_0 + \epsilon\hat{\rho}_1$$

In the event that the field is in a single-photon excited state, let $|\psi_1\rangle$ be the field state if the photon came from $s_1$ and $|\psi_2\rangle$ be the field state if the photon came from $s_2$. Since the sources are of equal brightness, the probability that the photon came from either one is $\frac{1}{2}$. Thus,

$$\hat{\rho}_1 = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$$

### B. Positive Operator Valued Measures (POVM)

To estimate the separation between the sources we need to make a measurement of some kind on the quantum state. In general, any quantum measurement is defined by Positive Operator Valued Measure (POVM). This is a set of measurement operators $\{\hat{\Pi}_k\}$ with outcomes $\{k\}$ such that they sum to the identity on the Hilbert space.

$$\sum_k \hat{\Pi}_k = \hat{I}$$

For a state $\hat{\rho}$ the probability of observing outcome $k$ is,

$$Pr(k) = tr(\hat{\rho}\hat{\Pi}_k)$$

The most common type of POVM is called a Von Neumann projective measurement, which consists of a set of projectors formed from a set of orthonormal basis vectors.

$$\{\hat{\Pi}_k = |\pi_k\rangle\langle\pi_k| : \langle\pi_i|\pi_j\rangle = \delta_{ij}\}$$

We will limit ourselves to Von Neumann projective measurements for the remainder of the lab. As it turns out, we will not need anything more sophisticated to resolve the two sources beyond the Rayleigh limit. One convenient property of Von Neumann projective measurements is that the observation probabilities reduce to diagonal matrix elements by invoking the cyclic property of the trace.

$$Pr(k) = tr(\hat{\rho}\hat{\Pi}_k) = tr(\hat{\rho}\,|\pi_k\rangle\langle\pi_k|) = \langle\pi_k|\hat{\rho}|\pi_k\rangle$$

## III. DIRECT DETECTION

Direct detection/imaging effectively registers photon arrivals spatially over a detector. Thus its POVM is defined as projectors on the position basis $|x\rangle$.

$$\text{POVM}_{DD} = \{\hat{\Pi}_x = |x\rangle\langle x|\}$$

For a shift-invariant PSF, we can define the single-photon excitation states in the position representation in the following way.

$$|\psi_1\rangle = \int_{-\infty}^{\infty} \psi\left(x - \frac{\theta}{2}\right)|x\rangle\, dx \quad (2)$$

$$|\psi_2\rangle = \int_{-\infty}^{\infty} \psi\left(x + \frac{\theta}{2}\right)|x\rangle\, dx \quad (3)$$

The position basis states $|x\rangle$ should be physically interpreted as a single-photon excitation at the location $x$. We can express this using the raising operator $|x\rangle = \hat{a}^{\dagger}\delta(x'-x)\,|\text{vac}\rangle$. Then the wavefunctions $\psi(x \mp \frac{\theta}{2})$ are understood to be a complex probability amplitudes over all possible localized excitations.

Given that we detected a photon, the probability that the arrived at position $x$ is given by,

$$p(x) = \langle x|\hat{\rho}_1|x\rangle = \frac{1}{2}|\,\langle x|\psi_1\rangle\,|^2 + \frac{1}{2}|\,\langle x|\psi_2\rangle\,|^2$$

**Problem 1:**
Write down the full expression for $p(x)$ - the probability of detecting a photon at location $x$ in direct imaging. Also, show that the states $|\psi_1\rangle\,, |\psi_2\rangle$ are not orthogonal and plot $|\,\langle\psi_1|\psi_2\rangle\,|^2$ as function of $\theta$. What does this say about the distinguishability of the two states? How would you expect the number of detected photons needed to accurately estimate the source separation to change with the separation distance itself?

*NOTE: The last part of this question wants you to think about why resolution limits exist at all. If the two states were perfectly distinguishable (orthogonal), would we be able to determine the source separation exactly?*

**Problem 2:**
With $\sigma = 1$, simulate direct detection of $10^5$ photons over the x-axis for source separations $\theta = [\sigma/10, \sigma/2, \sigma, 2\sigma, 10\sigma]$ by sampling from the distribution you found in Problem 1. Plot a histogram of the arrival statistics for each $\theta$. How does the number of peaks in the histogram change when $\theta > \sigma$ versus when $\theta < \sigma$?

*HINT: You may find matlab's* `normrnd` *and* `binornd` *functions useful.*

**Problem 3:**
Calculate the Maximum Likelihood Estimator (MLE) $\check{\theta}_{DD}$ of the source separation for each of the simulated direct detection measurements generated in Problem 2. Then, plot the fractional error $\frac{\check{\theta}_{DD}-\theta}{\theta}$ as a function of the source separation. What if we now increase the number of detected photons to $10^7$? How does the accuracy of the estimation change? Does this agree with your predictions about the photon requirements made in Problem 1?

*HINT: You may want to look into the Expectation Maximization algorithm described on Slide 19 of reference [6] as Gaussian mixtures do not in general have a closed form expressions for their parameter MLEs. You will need to implement an iterative method for finding the MLE.*

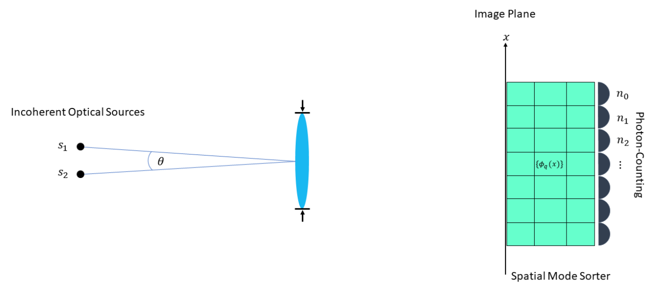


**FIG. 3:** A diagram of Spatial Mode Demultiplexing. The field at the image plane is decomposed into a desired modal basis with a spatial mode sorter. Then photon-counting is performed on each modal bin.

## IV. SPATIAL MODE DEMULTIPLEXING

Instead of measuring our field state in the position basis, we may choose to measure the field sate in any arbitrary orthogonal basis $\{|\phi_q\rangle\}$. This involves first defining a set of orthonormal modes $\{\phi_q(x)\}$ over which we quantize the field. A measurement in this basis amounts to observing a single-photon excitation in one of the modes $|\phi_q\rangle = \hat{a}_q^{\dagger}\,|\text{vac}\rangle$. The process of decomposing the field into a set of orthogonal modes and then counting photons in each mode is a measurement scheme known as Spatial Mode Demultiplexing (SPADE). We will see how using SPADE with the Hermite-Gauss (HG) modes allows us to better estimate the source separation compared to direct detection.

The $q^{th}$ HG mode is given by the function

$$\phi_q(x) = \frac{1}{(2\pi\sigma^2)^{1/4}}\frac{1}{\sqrt{2^q q!}}H_q\left(\frac{x}{\sigma\sqrt{2}}\right)e^{-\frac{x^2}{4\sigma^2}}$$

We can define the orthonormal HG basis $\{|\phi_q\rangle\}$ as the states given by,

$$|\phi_q\rangle = \int_{-\infty}^{\infty}\phi_q(x)\,|x\rangle\,dx$$

in the position representation. Hence the POVM for photon detection in the HG states becomes,

$$\text{POVM}_{HG} = \{\hat{\Pi}_q = |\phi_q\rangle\langle\phi_q|\}$$

The single-photon excitation states $|\psi_1\rangle$ and $|\psi_2\rangle$ can be represented in the HG basis as,

$$|\psi_1\rangle = \sum_{q=0}^{\infty} \frac{(-\theta/(4\sigma))^q}{\sqrt{q!}} e^{-\frac{1}{2}(\theta/(4\sigma))^2} |\phi_q\rangle \quad (4)$$

$$|\psi_2\rangle = \sum_{q=0}^{\infty} \frac{(+\theta/(4\sigma))^q}{\sqrt{q!}} e^{-\frac{1}{2}(\theta/(4\sigma))^2} |\phi_q\rangle \quad (5)$$

Interestingly, in this representation $|\psi_1\rangle$ and $|\psi_2\rangle$ take the form of coherent states over the HG modes with displacements $\mp\theta/(4\sigma)$ respectively. Intuitively, this can be understood as the quantum mechanical description of shifting the PSF off-axis. This interpretation is only possible because the $0^{th}$ HG mode is precisely our PSF, $|\phi_0\rangle = \int_{-\infty}^{\infty} dx\psi(x)\,|x\rangle$. Thus, the field state produced by shifted sources is just the displacement operator $\hat{\mathcal{D}}(\pm\theta/(4\sigma))\,|\phi_0\rangle$ applied to the $0^{th}$ HG mode.

The probability of measuring a single-photon excitation in the $q^{th}$ HG state is given by,

$$p[q] = \langle\phi_q|\hat{\rho}|\phi_q\rangle = \frac{1}{2}|\langle\phi_q|\psi_1\rangle|^2 + \frac{1}{2}|\langle\phi_q|\psi_1\rangle|^2$$

where square-brackets are used to denote a discrete probability distribution.

**Problem 4:**
Find an expression for the probability distribution $p[q]$. Do you recognize this distribution? What is the mean? What is the variance?

**Problem 5:**
Simulate the detection of $10^5$ photons under the HG SPADE measurement scheme by sampling from the distribution you found in Problem 4. Use the same choice of $\sigma$ and range of $\theta$'s as in Problem 2.

*HINT: You may find Matlab's* `poissrnd` *helpful.*

**Problem 6:**
Derive an expression of of the maximum likelihood estimator $\check{\theta}_{HG}$ for the source separation in terms of the maximum likelihood estimator for the mean of a Poisson distribution. Compute $\check{\theta}_{HG}$ for each of the simulated HG SPADE measurements generated in Problem 5. Then, plot the fractional error $|\frac{\check{\theta}_{HG}-\theta}{\theta}|$ as a function of the source separation. How does the error compare to the MLE error plot you made in Problem 3. In particular, how do the accuracy of the source separation estimates differ in the 'super-Rayleigh'regimes where $\theta > \sigma$ and the 'sub-Rayleigh' regimes $\theta < \sigma$ for either measurement scheme?

*HINT: You should find that the MLE*

$$\check{\theta}_{HG} = 4\sigma\sqrt{\check{Q}}$$

where $Q$ is the mean of the Poisson distribution and

$$\check{Q} = \frac{1}{N}\sum_{q=0}^{\infty} q n_q$$

is its MLE for a measurement in which $N$ photons were detected in total and $n_q$ was the number of photons detected in mode $q$.

## V. FISHER INFORMATION

As you have found, the SPADE measurement scheme in the HG modes manages to drastically outperform direct imaging for estimating the source separation in the sub-Rayleigh regime $\theta < \sigma$. But the HG modes were only one possible set of orthogonal modes that we could have chosen out of an infinite number of modal bases. For instance, instead of the Hermite-Gaussian functions $\phi_q(x)$, we could have chosen the complex exponentials $f_\omega(x) = e^{-i\omega x}$. In this case, our POVM would have amounted to measuring the Fourier spectrum of the scene. This begs an important question: "How can we quantify the efficacy of *any* POVM for estimating the source separation?"

The Fisher Information is one such metric for quantifying the quality of a POVM within the context of any parameter estimation problem, not just the two point source problem. Abstractly, the Fisher Information tells us how much information a probability distribution contains about the parameters that characterize the distribution. Practically, this means that the Fisher Information can tell us how much information a single detected photon contains about the parameters of the scene from which the photon was emitted. In the following sections we will define the Fisher Information and try to build some intuition for what it does and why it is useful.

### A. Fisher Information Definition

Like all information measures, the Fisher Information is defined in relation to a probability distribution. In general, this can be a multi-variable distribution with several distribution parameters. However, in this lab we will be restricting ourselves to single-variable single-parameter distributions $p(y;\theta)$ where $y$ is the variable and $\theta$ is a parameter of the distribution (not necessarily source separation). In this special case, the Fisher Information is defined to be the variance of the *score* - the derivative of

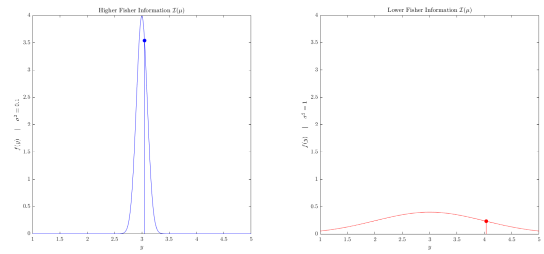


**FIG. 4:** Two gaussian distributions are shown with the same mean but different variances. We see that a sample from a narrow gaussian (blue stem) is more likely to reside close to the mean $\mu$ than a sample from a wide gaussian (red stem). As a result, the narrow gaussian has a higher Fisher Information on the the parameter $\mu$ compared to the wide gaussian.

the log-likelihood with respect to the parameter.

$$\mathcal{I}(\theta) = \mathbb{E}_p\left[\left(\frac{d}{d\theta}\ln(p)\right)^2\right]$$

### B. Example: Fisher Information of a Gaussian

Suppose we make a measurement of a random variable $Y$ that is normally distributed $Y \sim \mathcal{N}(\mu, \sigma^2)$. We don't know the parameter $\mu$, the mean of the distribution, but we do know the variance $\sigma^2$. How much does this random sample of $Y$ tell us about the mean $\mu$ of the distribution?

Before jumping into the math, lets think: if the gaussian is very narrow then we likely drew a sample near the mean. In this case, our sample contains a lot of information about the mean. Otherwise, if the gaussian is very wide, then the random sample doesn't say much about the mean since we likely drew it from somewhere further away. This is illustrated in figure 4. So we might expect the Fisher Information $\mathcal{I}(\mu)$ to somehow depend inversely on the variance $\sigma^2$. Recall that a gaussian is defined as,

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Indeed, if we work through the Fisher Information for $\mu$ in a gaussian distribution with known variance, we find

$$\begin{aligned} \ln(f) &= -\frac{(y-\mu)^2}{2\sigma^2} + const. \\ \frac{d}{d\mu}\ln(f) &= \frac{y-\mu}{\sigma^2} \end{aligned}$$

$$\begin{aligned} \mathcal{I}(\mu) &= \mathbb{E}_f\left[\left(\frac{d}{d\mu}\ln(f)\right)^2\right] \\ &= \frac{1}{\sigma^4}\mathbb{E}_f[(y-\mu)^2] \\ &= \frac{1}{\sigma^4}Var(Y) \\ &= \frac{1}{\sigma^2} \end{aligned}$$

As we intuited, the Fisher Information on the mean of a gaussian random variable is inversely proportional to the variance.

### C. Cramer-Rao Lower Bound

The inverse of the Fisher Information is the Cramer-Rao Lower Bound (CRLB). The CRLB states that for any unbiased estimator $\check{\theta}$ of a distribution parameter $\theta$, its variance is lower-bounded by

$$Var(\check{\theta}) \geq \frac{1}{N}\mathcal{I}^{-1}(\theta)$$

where $N$ is the number of samples (measurements) drawn from the distribution. A major goal of parameter estimation efforts is to design measurements with very low CRLBs. This way the estimators are more accurate and consistent.

### D. Quantum Fisher Information

For completeness, we provide the Quantum Fisher Information (QFI) proposed by Helstrom [1] for a single parameter, which is given by

$$QFI(\theta) = Re\{tr(\hat{\mathcal{L}}_\theta^2(\hat{\rho})\hat{\rho})\}$$

where $\hat{\mathcal{L}}_\theta$ is called the Symmetric Logarithmic Derivative (SLD) and satisfies the relation,

$$2\frac{d}{d\theta}\hat{\rho} = \hat{\mathcal{L}}_\theta\hat{\rho} + \hat{\rho}\hat{\mathcal{L}}_\theta$$

If we write the density operator in its eigenbasis,

$$\hat{\rho} = \sum_i D_i \left|e_i\right\rangle\!\left\langle e_i\right|$$

then the SLD can be expressed in closed form as,

$$\hat{\mathcal{L}}_\theta = \sum_{D_j+D_k\neq 0} \frac{2}{D_j+D_k}\left\langle e_j\right|\frac{\partial\hat{\rho}}{\partial\theta}\left|e_k\right\rangle$$

The QFI represents the maximum amount of information that we may extract about a parameter using *any* measurement scheme. In other words, the QFI defines a global information upper-bound (and hence the ultimate lower-bound on the estimator variance) allowed by the laws of physics. Crucially though, the QFI does not necessarily tell us which measurement scheme achieves this upper-bound.

The QFI for the source separation is,

$$\mathcal{I}(\theta) = \frac{1}{4\sigma^2}$$

which is derived in reference [5].

### E. Fisher Information for Direct Detection

The probability distribution over the x-axis for a photon arrival is given by the gaussian mixture,

$$p(x) = \frac{1}{2}\Big[\Big(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\frac{\theta}{2})^2/(2\sigma^2)} + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+\frac{\theta}{2})^2/(2\sigma^2)}\Big] \tag{6}$$

The Fisher Information of the source separation under direct detection can be written explicitly as

$$I_{DD}(\theta) = \int_{\infty}^{\infty} p(x)\left[\frac{d}{d\theta}(\ln(p(x))\right]^2 dx \tag{7}$$

$$= \int_{\infty}^{\infty} \frac{1}{p(x)}\left(\frac{dp(x)}{d\theta}\right)^2 dx \tag{8}$$

Taking the derivative of the gaussian mixture, we find

$$\frac{dp(x)}{d\theta} = \frac{1}{4\sigma^2}\left[\frac{(x-\theta/2)}{\sqrt{2\pi\sigma^2}}e^{-(x-\frac{\theta}{2})^2/(2\sigma^2)} \dots \right.$$
$$\left. - \frac{(x+\theta/2)}{\sqrt{2\pi\sigma^2}}e^{-(x+\frac{\theta}{2})^2/(2\sigma^2)}\right]$$
$$= \frac{1}{4\sigma^2}\left[\left(x-\frac{\theta}{2}\right)|\psi(x-\frac{\theta}{2})|^2 - \left(x+\frac{\theta}{2}\right)|\psi(x+\frac{\theta}{2})|^2\right]$$

where $\psi_+$

**Problem 7:**
Evaluate the Fisher Information under direct detection $I_{DD}(\theta)$ for $\theta$ in range $(0.1\sigma, 10\sigma)$ and plot its normalized version $I_{DD}(\theta)/I(\theta) = I_{DD}(\theta)/(1/4\sigma^2)$. Also plot the Fisher information under the Hermite-Guass SPADE measurement scheme.

Plot the Fisher information $I(\theta)$ as a function of of the source separation $\theta$ by numerically evaluating the integral for linearly spaced samples of $\theta$ over the interval $(0, 5\sigma]$. How does direct detection compare to SPADE in terms of Fisher Information when the separation is below and above the Rayleigh Limit?

*Hint: Evaluating the Fisher Information $\mathcal{I}_{DD}(\theta)$ analytically for this gaussian mixture probability is intractable. We can do it numerically though, just make sure you finely discretize the x-axis in the regions within $5\sigma$ of $\pm\theta/2$ for accurate results.*

### F. Fisher Information for HG SPADE

The Fisher Information of the source separation under HG SPADE measurements can be written explicitly as

$$I_{HG}(\theta) = \sum_{q=0}^{\infty} p[q]\left[\frac{d}{d\theta}\ln(p[q])\right]^2$$

The probability of detecting a photon in the $q^{th}$ HG mode is given by the Poisson distribution,

$$p(q) = \frac{Q^q}{q!}e^{-Q} \qquad Q \equiv \frac{\theta^2}{16\sigma^2} \tag{9}$$

where $Q$ is the mean of the Poisson distribution. From this discrete distribution we can find,

$$I_{HG}(\theta) = \frac{1}{4\sigma^2} \tag{10}$$

Remarkably, we find that the Fisher Information for the separation parameter is independent of $\theta$. This means that the Fisher Information of the source separation remains constant for arbitrarily small separation distances under the SPADE measurement scheme.

**Problem 8:**
Derive the Fisher Information for the HG SPADE measurements $I_{HG}(\theta)$. Show that it is equal to the Quantum Fisher Information $I(\theta)$ proving that the Hermite-Gauss functions are the optimal set of modes in which to decompose the field for estimating source separation.

### ACKNOWLEDGMENTS

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## Appendix A: Solutions to Lab Problems

### 1. Problem 1 Solution

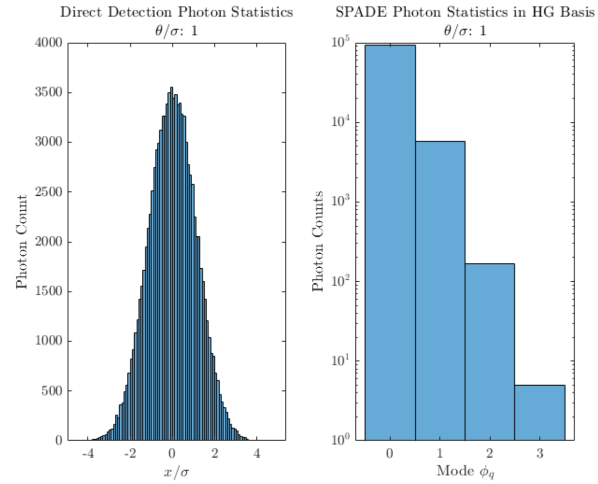
The probability distribution over the image plane under direct detection is,

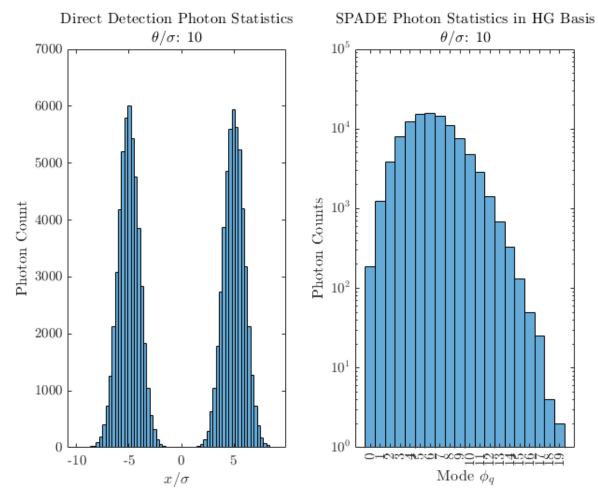
$$p(x) = \frac{1}{2}\Big[\big(\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\frac{\theta}{2})^2/(2\sigma^2)} + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+\frac{\theta}{2})^2/(2\sigma^2)}\Big] \tag{A1}$$

The inner product between the field states produced by $s_1$ and $s_2$ is,

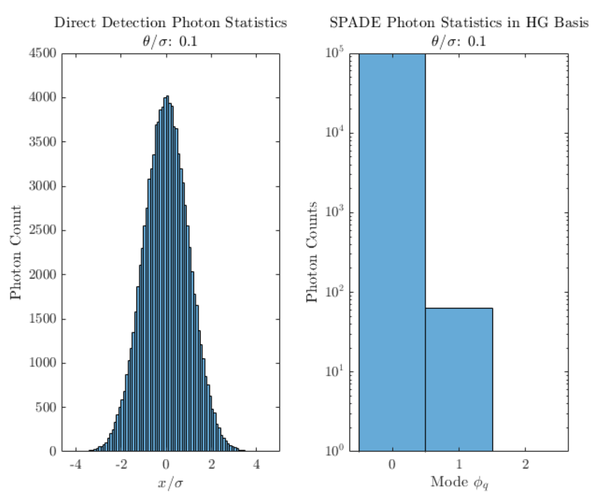
$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= \int_{-\infty}^{\infty} \psi(x-\theta/2)\psi(x+\theta/2)dx \\ &= \frac{1}{\sqrt{2\pi^2}}\int_{-\infty}^{\infty} e^{-\frac{(x-\theta/2)^2}{4\sigma^2}}e^{-\frac{(x+\theta/2)^2}{4\sigma^2}} \\ &= e^{-\frac{\theta^2}{8\sigma^2}}dx \end{aligned}$$

Note that as $\theta \to \infty$ the inner product goes to 0. Hence the sources are perfectly distinguishable from single measurement if they are infinitely far away from each other. This might be confusing as $\theta$ was defined to be an angle, so its domain is $[0, 2\pi)$. But really we've defined $\theta$ in the paraxial regime which uses the small-angle approximation. So in fact, $\theta$ actually corresponds to a distance ratio which has domain $[0, \infty)$. This ratio is the source separation distance over the distance of the object plane to the lens.

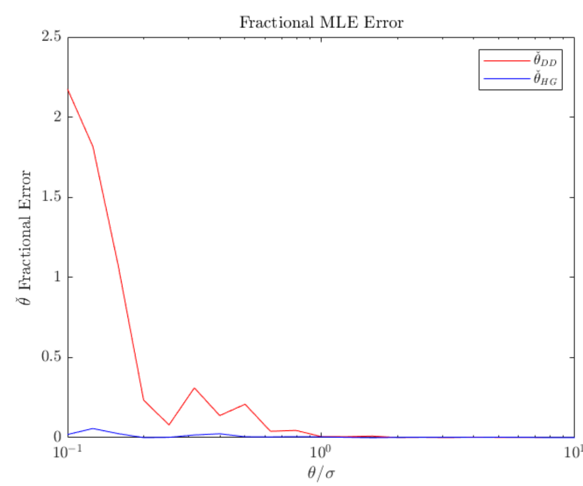



### 2. Problem 2/5 Solution



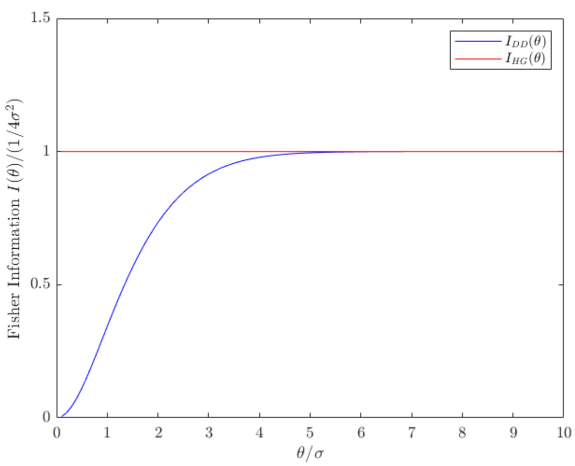

### 3. Problem 3/6 Solution



### 4. Problem 4 Solution

$$p[q] = \frac{Q^q}{q!}e^{-Q} \qquad Q \equiv \frac{\theta^2}{16\sigma^2}$$

### 5. Problem 7 Solution




### 6. Problem 8 Solution

$$p(q) = \frac{Q^q}{q!}e^{-Q} \qquad Q \equiv \frac{\theta^2}{16\sigma^2} \tag{A2}$$

where $Q$ is the mean of the Poisson distribution.

$$I_{HG}(\theta) = \sum_{q=0}^{\infty} p(q)\left[\frac{d}{d\theta}(\ln p(q))\right]^2$$

$$\begin{aligned}
\frac{d}{d\theta}(\ln p(q)) &= \frac{d}{d\theta}(-Q + q\ln(Q) - \ln(q!)) \\
&= -Q' + \frac{q}{Q}Q' \\
&= (\frac{q}{Q} - 1)\frac{2}{16\sigma^2}\theta \\
&= 2(q-Q)/\theta
\end{aligned}$$

$$\begin{aligned}
I_{HG}(\theta) &= \sum_{q=0}^{\infty} p(q)\left[\frac{d}{d\theta}(\ln p(q))\right]^2 \\
&= \sum_{q=0}^{\infty} p(q)\left[2(q-Q)/\theta\right]^2 \\
&= \frac{4}{\theta^2}\sum_{q=0}^{\infty} p(q)(q-Q)^2 \\
&= \frac{4}{\theta^2}Var(q) \\
&= \frac{4}{\theta^2}Q \\
&= \frac{1}{4\sigma^2}
\end{aligned}$$

$$I_{HG}(\theta) = \frac{1}{4\sigma^2} \tag{A3}$$

## REFERENCES

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