Deriving the Quantum Hamiltonian for Reflective Optomechanical Membrane

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1 Problem Statement

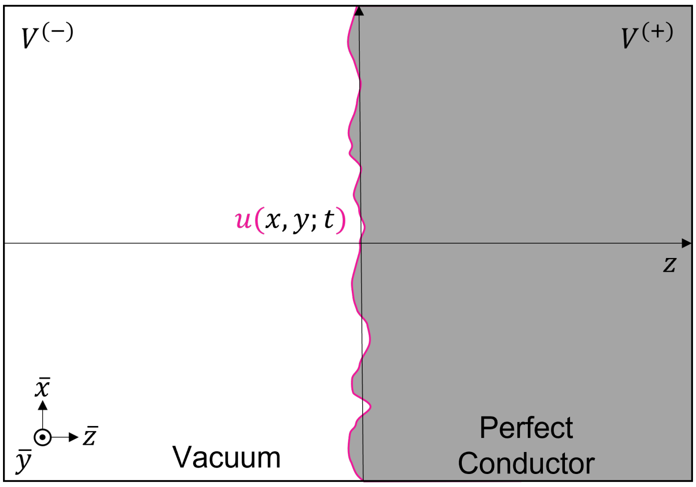

Figure 1: A depiction of the half-space construction for a time-varying mirror.

Consider a half-space defined by a compliant perfect electrical conductor (PEC) where the boundary of the half-space (surface of the conductor) is defined by the function $u(x, y, t)$ as shown in figure 1. Physically, the boundary surface is a mirror with unit reflectivity at all wavelengths. Because the surface is compliant, our task is to define a quantum Hamiltonian for the total energy in the EM field and the vibrating surface. We will use this half-space model as the first step towards ultimately defining the quantum Hamiltonian for a compliant 2D membrane.

2 Boundary Conditions

All of electrodynamics involves the vector fields,

- **E** - Electric Field [Volt/meter]
- **H** - Magnetic Field [Ampere/meter]
- **P** - Polarization [Coulomb/meter]
- **M** - Magnetization [Amperes/meter]
- $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ - Electric Displacement
- $\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$ - Magnetic Induction

which we define at the outset to eliminate any ambiguity that might arise due to differences in definitional conventions. For these field, Maxwell's boundary conditions at a surface are given by equation 1

$$D_{\perp}^{(+)}(\mathbf{r}_s, t) - D_{\perp}^{(-)}(\mathbf{r}_s, t) = \sigma_{surf}(\mathbf{r}_s, t) \tag{1a}$$

$$\mathbf{H}_{\parallel}^{(+)}(\mathbf{r}_s, t) - \mathbf{H}_{\parallel}^{(-)}(\mathbf{r}_s, t) = \mathbf{J}_{surf}(\mathbf{r}_s, t) \times \bar{\mathbf{n}}(\mathbf{r}_s, t) \tag{1b}$$

$$\mathbf{E}_{\parallel}^{(+)}(\mathbf{r}_s, t) = \mathbf{E}_{\parallel}^{(-)}(\mathbf{r}_s, t) \tag{1c}$$

$$B_{\perp}^{(+)}(\mathbf{r}_s, t) = B_{\perp}^{(-)}(\mathbf{r}_s, t) \tag{1d}$$

where $\mathbf{r}_s$ is a point on the surface and $(-)/(+)$ denotes the field immediately to the left/right of the half-space surface. σ_{surf} and $\mathbf{J}_{surf}$ are the surface charge and current densities respectively induced by the incident field while $\bar{\mathbf{n}}$ is the surface normal unit vector. By convention, we choose the normal vector to have a positive projection along the z-axis (the surface normal points primarily to the right in figure 1). The EM field inside the volume $V^{(+)}$ of the PEC is zero. Therefore, we have ($\mathbf{E}^{(+)} = \mathbf{H}^{(+)} = 0$). The electric and magnetic polarizations, $\mathbf{P}$ and $\mathbf{M}$, are also zero inside the PEC as dipoles cannot be induced. Therefore the displacement fields reduce to $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ respectively. In all, we find the much simplified boundary conditions for the half-space,

$$E_{\perp}^{(-)}(\mathbf{r}_s, t) = -\frac{1}{\epsilon_0}\sigma_{surf}(\mathbf{r}_s, t) \tag{2a}$$

$$\mathbf{H}_{\parallel}^{(-)}(\mathbf{r}_s, t) = -\mathbf{J}_{surf}(\mathbf{r}_s, t) \times \bar{\mathbf{n}}(\mathbf{r}_s, t) \tag{2b}$$

$$\mathbf{E}_{\parallel}^{(-)}(\mathbf{r}_s, t) = 0 \tag{2c}$$

$$H_{\perp}^{(-)}(\mathbf{r}_s, t) = 0 \tag{2d}$$

3 Quantum Hamiltonian

The complete Hamiltonian for this closed system is defined as the total energy of the electromagnetic field in the volume of the half-space plus the mechanical energy of the oscillating boundary $\hat{H}_{tot} = \hat{H}_V + \hat{H}_S$. The total Hamiltonian is given explicitly as,

$$\hat{H}_{tot} = \frac{1}{2}\int_{\hat{V}^{(-)}} \epsilon_0 \hat{\mathbf{E}}^2 + \mu_0 \hat{\mathbf{H}}^2 dV + \sum_{nm} \hbar\omega_{nm}\left(\hat{b}_{nm}^{\dagger}\hat{b}_{nm} + \frac{1}{2}\right) \tag{3}$$

where $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ are the electric and magnetic field operators and $\hat{b}_{nm}^{\dagger}, \hat{b}_{nm}$ are the creation and annihilation operators for the mechanical modes of the boundary surface with natural frequency ω_{nm}. We have used the square of the vector field operators as a shorthand to indicate the complex dot product, namely $\hat{\mathbf{E}}^2 = \hat{\mathbf{E}} \cdot \hat{\mathbf{E}} = \hat{E}_x^{\dagger}\hat{E}_x + \hat{E}_y^{\dagger}\hat{E}_y + \hat{E}_z^{\dagger}\hat{E}_z$. Note that the domain of integration is also an operator quantity. This is because the half-space boundary $\hat{u}(x, y, t)$ is considered to be a dynamical quantum variable. A more standard way of writing the Hamiltonian for the EM field in free-space replaces $\mathbf{H}$ with $\frac{1}{\mu_0}\mathbf{B}$ so that

$$\hat{H}_V = \frac{1}{2}\int_{\hat{V}^{(-)}} \epsilon_0 \hat{\mathbf{E}}^2 + \frac{1}{\mu_0}\hat{\mathbf{B}}^2 dV = \frac{1}{2}\int_{\mathbb{R}^2}\int_{-\infty}^{\hat{u}(x,y,t)} \epsilon_0 \hat{\mathbf{E}}^2 + \frac{1}{\mu_0}\hat{\mathbf{B}}^2 dz dx dy \tag{4}$$

To properly quantize the EM field and introduce creation and annihilation operators we must define a complete orthonormal basis of spatio-temporal modes $\{\phi_\theta(x, y, z, t)\}$ that satisfy the wave equation and prescribed boundary conditions - here θ represents a generalized (continuous or discrete) index. Any physically realizable EM field state can then be described as a tensor product state over independent excitation states of each mode. We follow the orthodox approach of using a standard plane-wave decomposition of the field assuming a flat PEC surface. The effect of undulations in the boundary are subsequently accounted for with a perturbation term that is valid for small surface displacements. The caveat is that the plane wave modes satisfying the boundary conditions for a flat surface do not in general satisfy the boundary conditions for an undulating one. Nevertheless the boundary conditions are approximately satisfied in the regime of small surface displacements which is of primary relevance in optomechanics.

4 Plane-Wave Expansion

For small surface displacements we can expand the Hamiltonian into two terms by breaking up the bounds of integration along the z-axis.

$$\hat{H}_V \approx \frac{1}{2}\int_{\mathbb{R}^2}\int_{-\infty}^{0} \epsilon_0 \hat{\mathbf{E}}^2 + \frac{1}{\mu_0}\hat{\mathbf{B}}^2 dz dx dy + \frac{1}{2}\int_{\mathbb{R}^2} \hat{u}(x, y, t)\left[\epsilon_0 \hat{\mathbf{E}}^2 + \frac{1}{\mu_0}\hat{\mathbf{B}}^2\right]_{z=0} dx dy \tag{5}$$

The first involves the EM field energy in the half-space defined by a flat planar mirror while the second is an interaction energy between the pliable membrane surface and the EM field. For a derivation of this approximation see appendix A. In brief, we express these two terms as,

$$\hat{H}_V \approx \hat{H}_{EM} + \hat{H}_{int} \tag{6}$$

Using a plane wave expansion for a flat rigid boundary, [1] finds (as we would expect) that the first term is simply the sum of the energies over independent harmonic oscillators

$$
\begin{aligned}
\hat{H}_{EM} &= \frac{1}{2}\sum_{j=s,p}\int_{\mathbb{R}^2} d\mathbf{k}_\parallel \int_0^\infty dk_z \hbar\omega[\hat{a}_j^\dagger(\mathbf{k})\hat{a}_j(\mathbf{k}) + \hat{a}_j(\mathbf{k})\hat{a}_j^\dagger(\mathbf{k})] \\
&= \sum_{j=s,p}\int_{\mathbb{R}^2} d\mathbf{k}_\parallel \int_0^\infty dk_z \hbar\omega\left(\hat{a}_j^\dagger(\mathbf{k})\hat{a}_j(\mathbf{k}) + \frac{1}{2}\right)
\end{aligned}
$$

where $\hat{a}_j^\dagger(\mathbf{k}), \hat{a}_j(\mathbf{k})$ are creation and annihilation operators for the plane-wave modes in equation 7 with orthogonal s and p polarizations.

$$\boldsymbol{\phi}_s(\mathbf{r}, t; \mathbf{k}) = i(\bar{\mathbf{k}}_\parallel \times \bar{\mathbf{z}})\sin(k_z z)\exp\big(i(\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel - \omega t)\big) \tag{7a}$$

$$\boldsymbol{\phi}_p(\mathbf{r}, t; \mathbf{k}) = [i\bar{\mathbf{k}}_\parallel \frac{k_z}{k}\sin(k_z z) - \bar{\mathbf{z}}\frac{k_\parallel}{k}\cos(k_z z)]\exp\big(i(\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel - \omega t)\big) \tag{7b}$$

Here we use the overbar notation $\bar{\mathbf{u}}$ to denote unit vectors. We have also made definitions $\mathbf{r}_\parallel = (x, y)$, $\mathbf{k}_\parallel = (k_x, k_y)$, and $\omega^2 = c^2(k_\parallel^2 + k_z^2)$. The plane-wave operators have units of $\sqrt{\text{volume}}$ and satisfy the commutation relation,

$$[\hat{a}_i(\mathbf{k}), \hat{a}_j^\dagger(\mathbf{k}')] = \delta_{ij}\delta(\mathbf{k} - \mathbf{k}') \qquad i, j = s, p \tag{8}$$

Now we turn to writing down the expression for $\hat{H}_{int}$ in the plane wave basis by following suit with [1]. The field operators $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ can be expressed in terms of the plane-wave operators for waves propagating in the positive and negative directions.

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \hat{\mathbf{E}}^+(\mathbf{r}, t) + \hat{\mathbf{E}}^-(\mathbf{r}, t)$$

The positive and negative frequency parts of the field operator are Hermitian conjugates of each other $\hat{\mathbf{E}}^+ = (\hat{\mathbf{E}}^-)^\dagger$. There is a subtle distinction in notation here: superscripts $(\pm)$ with parentheses denote the field to the right and left of the half-space boundary while we use supercripts $\pm$ without parentheses to denote the field expansion in right and left propagating plane waves. We may further decompose the field into s and p polarization components. In particular the s polarization is *perpendicular* to the plane containing the wavevector $\mathbf{k}$ and the z-axis, while the p polarization is *parallel* to the plane containing the wavevector $\mathbf{k}$ and the z-axis.

$$\hat{\mathbf{E}}^+(\mathbf{r}, t) = \hat{\mathbf{E}}_s^+(\mathbf{r}, t) + \hat{\mathbf{E}}_p^+(\mathbf{r}, t) \tag{9}$$

If we assume (for now) that the boundary surface is flat, then for $z > 0$ the EM modes are defined to be zero categorically due to the presence of the PEC. For $z \le 0$ the plane-wave decomposition of the electric field operators is,

$$\hat{\mathbf{E}}_s^+(\mathbf{r}, t) = i\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0}}\hat{a}_s(\mathbf{k})(\bar{\mathbf{k}}_\parallel \times \bar{\mathbf{z}})\sin(k_z z)\exp\big(i(\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel - \omega t)\big) \tag{10a}$$

$$\hat{\mathbf{E}}_p^+(\mathbf{r}, t) = \int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0}}\hat{a}_p(\mathbf{k})[i\bar{\mathbf{k}}_\parallel \frac{k_z}{k}\sin(k_z z) - \bar{\mathbf{z}}\frac{k_\parallel}{k}\cos(k_z z)]\exp\big(i(\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel - \omega t)\big) \tag{10b}$$

Note in equations 10a and 10b that the E-field component parallel to the boundary is zero when $z = 0$ as required by the boundary conditions due to the $\sin(k_z z)$ factor. Only the p-polarization term has a non-zero component at the boundary which is perpendicular to the flat surface. In turn, the plane-wave decomposition for the magnetic field operators is,

$$\hat{\mathbf{B}}_s^+(\mathbf{r}, t) = \int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0 c^2}}\hat{a}_s(\mathbf{k})(\bar{\mathbf{k}}_\parallel \frac{k_z}{k}\cos(k_z z) - i\bar{\mathbf{z}}\frac{k_\parallel}{k}\sin(k_z z))\exp\big(i(\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel - \omega t)\big) \tag{11a}$$

$$\hat{\mathbf{B}}_p^+(\mathbf{r}, t) = -\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0 c^2}}\hat{a}_p(\mathbf{k})(\bar{\mathbf{k}}_\parallel \times \bar{\mathbf{z}})\cos(k_z z)\exp\big(i(\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel - \omega t)\big) \tag{11b}$$

where the optical modes for the magnetic field have been phase shifted in order to satisfy boundary conditions. Here when $z = 0$ the perpendicular components of the magnetic field go to zero (again because of the $\sin(k_z z)$ factor). To determine $\hat{H}_{int}$ we evaluate both electric and magnetic fields at the plane $z = 0$.

$$\hat{\mathbf{E}}_s^+(\mathbf{r}_\parallel, t)|_{z=0} = 0 \tag{12a}$$

$$\hat{\mathbf{E}}_p^+(\mathbf{r}_\parallel, t)|_{z=0} = -\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0}}\hat{a}_p(\mathbf{k})\frac{k_\parallel}{k}\exp\big(i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel - \omega t)\big)\bar{\mathbf{z}} \tag{12b}$$

$$\hat{\mathbf{B}}_s^+(\mathbf{r}_\parallel, t)|_{z=0} = +\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0 c^2}}\hat{a}_s(\mathbf{k})\frac{k_z}{k}\exp\big(i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel - \omega t)\big)\bar{\mathbf{k}}_\parallel \tag{12c}$$

$$\hat{\mathbf{B}}_p^+(\mathbf{r}_\parallel, t)|_{z=0} = -\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z \sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0 c^2}}\hat{a}_p(\mathbf{k})\exp\big(i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel - \omega t)\big)(\bar{\mathbf{k}}_\parallel \times \bar{\mathbf{z}}) \tag{12d}$$

4.1 Aside: Surface Charge Density and Surface Current Density Operators

Let us briefly take stock of the equations in 12. Comparing these to the half-space boundary conditions in equation 2 we see they can be related to the surface charge density and surface current density (now in the form of operators). It is precisely the field at the boundary that gives rise to these sources (charge and current). Moreover, the self-interaction of the field with free charge carriers is what induces a radiation pressure on the mirror via the Lorentz Force.

$$\hat{\sigma}_{surf}(\mathbf{r}_\parallel, t) = -\epsilon_0\hat{E}_\perp = -\epsilon_0[\mathbf{E}_p^+ + \mathbf{E}_p^-]_{z=0}\cdot\bar{\mathbf{z}} \tag{13a}$$

$$= -\epsilon_0\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z\sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0}}\frac{k_\parallel}{k}[\hat{a}_p(\mathbf{k})e^{i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel-\omega t)} + \hat{a}_p^\dagger(\mathbf{k})e^{-i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel-\omega t)}] \tag{13b}$$

$$\hat{\mathbf{J}}_{surf}(\mathbf{r}_\parallel, t) = \frac{1}{\mu_0}\mathbf{B}_\parallel \times \bar{\mathbf{z}} = \frac{1}{\mu_0}[\mathbf{B}_s^+ + \mathbf{B}_s^- + \mathbf{B}_p^+ + \mathbf{B}_p^-]_{z=0}\times\bar{\mathbf{z}} \tag{14a}$$

$$\begin{aligned} = \frac{1}{\mu_0}\int_{\mathbb{R}^2} d^2k_\parallel \int_0^\infty dk_z\sqrt{\frac{\hbar\omega}{4\pi^3\epsilon_0 c^2}}\bigg(&\frac{k_z}{k_\parallel}[\hat{a}_s(\mathbf{k})e^{i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel-\omega t)} + \hat{a}_s^\dagger(\mathbf{k})e^{-i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel-\omega t)}](\hat{\mathbf{k}}_\parallel\times\hat{\mathbf{z}}) \\ &+[\hat{a}_p(\mathbf{k})e^{i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel-\omega t)} + \hat{a}_p^\dagger(\mathbf{k})e^{-i(\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel-\omega t)}]\bar{\mathbf{k}}_\parallel\bigg)\end{aligned} \tag{14b}$$

The Lorentz Force acting on the Boundary is given by,

$$\hat{\mathbf{F}}(\mathbf{r}_\parallel, t) = \hat{\sigma}_{surf}(\mathbf{r}_\parallel, t)\hat{\mathbf{E}}(\mathbf{r}_\parallel, t) + \frac{1}{\mu_0}\hat{\mathbf{J}}_{surf}(\mathbf{r}_\parallel, t)\times\hat{\mathbf{B}}(\mathbf{r}_\parallel, t) \tag{15}$$

Recall that the operator $\hat{u}$ corresponding to the PEC surface profile can be described through an expansion into its orthonormal mechanical modes $\psi_{nm}(\mathbf{r}_\parallel)$ where the mode amplitudes have been promoted to operators $\hat{c}_{nm}(t)$.

$$\hat{u}(\mathbf{r}_\parallel, t) = \sum_{nm}\hat{c}_{nm}(t)\psi_{nm}(\mathbf{r}_\parallel) \tag{16}$$

Following from the classical solutions for the mode amplitude, we may expand the mechanical mode amplitudes into quadratures via,

$$\begin{aligned}\hat{c}_{nm}(t) &= \hat{X}_{nm}\cos(\omega_{nm}t) + \hat{Y}_{nm}\sin(\omega_{nm}t)\\ &= [\hat{b}_{nm} + \hat{b}_{nm}^\dagger]\cos(\omega_{nm}t) - i[\hat{b}_{nm} - \hat{b}_{nm}^\dagger]\sin(\omega_{nm}t)\\ &= \hat{b}_{nm}e^{-i\omega_{nm}t} + \hat{b}_{nm}^\dagger e^{i\omega_{nm}t}\end{aligned}$$

where $\hat{b}, \hat{b}^\dagger$ are implicity evaluated at $t = 0$. Thus the mechanical mode creation/annihilation operators come coupled to a time dependent oscillatory term. This oscillatory term has implications for optical frequency detuning (side-bands) associated with a reflected light beam. With this the interaction Hamiltonian can be written as the sum of interaction energies for each mechanical mode

$$\hat{H}_{int} = \sum_{nm}\hat{H}_{nm} = \frac{1}{2}\sum_{nm}\hat{c}_{nm}(t)\int_{\mathbb{R}^2} d\mathbf{r}_\parallel\ \psi_{nm}(\mathbf{r}_\parallel)\left[\epsilon_0\hat{\mathbf{E}}^2 + \frac{1}{\mu_0}\hat{\mathbf{B}}^2\right]_{z=0} \tag{17}$$

where $\hat{H}_{nm}$ is the interaction energy with the nm^{th} mechanical mode. To determine these interaction energy operators, we will use the equations for the field operators evaluated at $z = 0$ found in equation 12. The square of the E-field at the boundary expands to,

$$\hat{\mathbf{E}}\cdot\hat{\mathbf{E}}|_{z=0} = \left[\hat{E}_p^-\hat{E}_p^- + \hat{E}_p^-\hat{E}_p^+ + \hat{E}_p^+\hat{E}_p^- + \hat{E}_p^+\hat{E}_p^+\right]_{z=0}$$

while the square of the B-Field at the boundary involves many terms which we collect into a matrix for convenience.

$$\hat{\mathbf{B}}\cdot\hat{\mathbf{B}}|_{z=0} = \texttt{sum}\begin{bmatrix} \hat{\mathbf{B}}_s^-\cdot\hat{\mathbf{B}}_s^- & \hat{\mathbf{B}}_s^-\cdot\hat{\mathbf{B}}_s^+ & \hat{\mathbf{B}}_s^-\cdot\hat{\mathbf{B}}_p^- & \hat{\mathbf{B}}_s^-\cdot\hat{\mathbf{B}}_p^+ \\ \hat{\mathbf{B}}_s^+\cdot\hat{\mathbf{B}}_s^- & \hat{\mathbf{B}}_s^+\cdot\hat{\mathbf{B}}_s^+ & \hat{\mathbf{B}}_s^+\cdot\hat{\mathbf{B}}_p^- & \hat{\mathbf{B}}_s^+\cdot\hat{\mathbf{B}}_p^+ \\ \hat{\mathbf{B}}_p^-\cdot\hat{\mathbf{B}}_s^- & \hat{\mathbf{B}}_p^-\cdot\hat{\mathbf{B}}_s^+ & \hat{\mathbf{B}}_p^-\cdot\hat{\mathbf{B}}_p^- & \hat{\mathbf{B}}_p^-\cdot\hat{\mathbf{B}}_p^+ \\ \hat{\mathbf{B}}_p^+\cdot\hat{\mathbf{B}}_s^- & \hat{\mathbf{B}}_p^+\cdot\hat{\mathbf{B}}_s^+ & \hat{\mathbf{B}}_p^+\cdot\hat{\mathbf{B}}_p^- & \hat{\mathbf{B}}_p^+\cdot\hat{\mathbf{B}}_p^+ \end{bmatrix}_{z=0}$$

Note that we do not denote the magnetic field dot product as a scalar immediately simply as a reminder that the plane-wave expansion of the magnetic field involves a $\mathbf{k}$-dependent unit vector. We also briefly note that cross-terms involving operators of different polarizations commute (proof in Appendix B). A sensible way to break up the interaction Hamiltonians is by collecting quadratic terms of joint polarization subscripts ss, sp (or ps), and pp.

$$\hat{H}_{nm} = \hat{H}_{nm}^{(ss)} + \hat{H}_{nm}^{(sp)} + \hat{H}_{nm}^{(pp)}$$

which, after some algebra, are

$$\begin{aligned}\hat{H}_{nm}^{(ss)} = \hat{c}_{nm}(t)\left(\frac{\hbar}{8\pi^2}\right)\int_K\int_{K'} d\mathbf{k}d\mathbf{k}'\left[\sqrt{\omega\omega'}\frac{k_z k_z'}{kk'}(\bar{\mathbf{k}}_\parallel\cdot\bar{\mathbf{k}}_\parallel')\right]\Big\{&\hat{a}_s(\mathbf{k})\hat{a}_s(\mathbf{k}')e^{-i(\omega+\omega')t}\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel+\mathbf{k}_\parallel') + H.C. \\ &+ \hat{a}_s(\mathbf{k})\hat{a}_s^\dagger(\mathbf{k}')e^{-i(\omega-\omega')t}\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel-\mathbf{k}_\parallel') \\ &+ \hat{a}_s^\dagger(\mathbf{k})\hat{a}_s(\mathbf{k}')e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel-\mathbf{k}_\parallel')\Big\}\end{aligned} \tag{18}$$

$$\begin{aligned}\hat{H}_{nm}^{(sp)} = \hat{c}_{nm}(t)\left(\frac{\hbar}{8\pi^2}\right)\int_K\int_{K'} d\mathbf{k}d\mathbf{k}'\left[\sqrt{\omega\omega'}\frac{k_z'}{k'}(\bar{\mathbf{k}}_\parallel\times\bar{\mathbf{k}}_\parallel')\cdot(-\bar{\mathbf{z}})\right]\Big\{&\hat{a}_s(\mathbf{k})\hat{a}_p(\mathbf{k}')e^{-i(\omega+\omega')t}\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel+\mathbf{k}_\parallel') + H.C. \\ &+ \hat{a}_s(\mathbf{k})\hat{a}_p^\dagger(\mathbf{k}')e^{-i(\omega-\omega')t}\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel-\mathbf{k}_\parallel') \\ &+ \hat{a}_s^\dagger(\mathbf{k})\hat{a}_p(\mathbf{k}')e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel-\mathbf{k}_\parallel')\Big\}\end{aligned} \tag{19}$$

$$\begin{aligned}\hat{H}_{nm}^{(pp)} = \hat{c}_{nm}(t)\left(\frac{\hbar}{8\pi^2}\right)\int_K\int_{K'} d\mathbf{k}d\mathbf{k}'\left[\sqrt{\omega\omega'}\left(\bar{\mathbf{k}}_\parallel\cdot\bar{\mathbf{k}}_\parallel' + \frac{k_\parallel k_\parallel'}{kk'}\right)\right]\Big\{&\hat{a}_p(\mathbf{k})\hat{a}_p(\mathbf{k}')e^{-i(\omega+\omega')t}\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel+\mathbf{k}_\parallel') + H.C. \\ &+ \hat{a}_p(\mathbf{k})\hat{a}_p^\dagger(\mathbf{k}')e^{-i(\omega-\omega')t}\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel-\mathbf{k}_\parallel') \\ &+ \hat{a}_p^\dagger(\mathbf{k})\hat{a}_p(\mathbf{k}')e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel-\mathbf{k}_\parallel')\Big\}\end{aligned} \tag{20}$$

where the integration domains K, K' are a shorthand for the space $\mathbb{R}^2\otimes(\mathbb{R}\geq 0)$ used before. We see that each of the three polarization terms in the interaction Hamiltonian have similar forms up to some weighting function. These functions can be understood as the *effective* frequency born from the interaction of two plane-waves and the geometry of their respective k-vectors.

$$\Omega^{(ss)}(\mathbf{k},\mathbf{k}') \equiv \sqrt{\omega\omega'}g^{(ss)}(\mathbf{k},\mathbf{k}') = \sqrt{\omega\omega'}\left[\frac{k_z k_z'}{kk'}(\bar{\mathbf{k}}_\parallel\cdot\bar{\mathbf{k}}_\parallel')\right] \tag{21a}$$

$$\Omega^{(sp)}(\mathbf{k},\mathbf{k}') \equiv \sqrt{\omega\omega'}g^{(sp)}(\mathbf{k},\mathbf{k}') = \sqrt{\omega\omega'}\left[\frac{k_z'}{k'}(\bar{\mathbf{k}}_\parallel\times\bar{\mathbf{k}}_\parallel')\cdot(-\bar{\mathbf{z}})\right] \tag{21b}$$

$$\Omega^{(pp)}(\mathbf{k},\mathbf{k}') \equiv \sqrt{\omega\omega'}g^{(pp)}(\mathbf{k},\mathbf{k}') = \sqrt{\omega\omega'}\left[\bar{\mathbf{k}}_\parallel\cdot\bar{\mathbf{k}}_\parallel' + \frac{k_\parallel k_\parallel'}{kk'}\right] \tag{21c}$$

The interaction Hamiltonian ostensibly leads to the radiation pressure force found in [1], however a new physical process has appeared. Namely, for a dynamic surface the interaction between plane waves is also mediated by the Fourier Transform of the mechanical modes $\tilde{\psi}_{nm}$, which we have defined using the unitary definition of the 2D FT.

$$\tilde{\psi}_{nm}(\mathbf{k}_\parallel) = \mathcal{F}_{\mathbf{k}_\parallel}\{\psi_{nm}\} = \frac{1}{2\pi}\int_{\mathbb{R}^2} d^2\mathbf{r}_\parallel\ \psi_{nm}(\mathbf{r}_\parallel)e^{-i\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel} \tag{22}$$

These functions have units of [area] since ψ_{nm} is dimensionless (in order for $\hat{c}_{nm}$ to have units of [length]). For convenience, we write each polarization pair in the interaction Hamiltonian generally with superscript indices,

$$\boxed{\begin{aligned}\hat{H}^{(\eta\nu)}_{nm} = \hat{c}_{nm}(t)\left(\frac{\hbar}{8\pi^2}\right)\int_K\int_{K'} d\mathbf{k}d\mathbf{k}'\Omega^{(\eta\nu)}(\mathbf{k},\mathbf{k}')\Big\{&\hat{a}_\eta(\mathbf{k})\hat{a}_\nu(\mathbf{k}')e^{-i(\omega+\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)+H.C.\\ &+\hat{a}_\eta(\mathbf{k})\hat{a}^\dagger_\nu(\mathbf{k}')e^{-i(\omega-\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)\\ &+\hat{a}^\dagger_\eta(\mathbf{k})\hat{a}_\nu(\mathbf{k}')e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)\Big\}\end{aligned}} \tag{23}$$

where $(\eta\nu) \in \{(ss), (sp), (pp)\}$.

4.2 Two-Mode Coupling from Interaction Hamiltonian

The Hamiltonian for a beam-splitter with transmission coefficient $\eta \in [0, 1)$ and phase $\phi \in [0, 2\pi)$ is given by

$$\hat{H}_{BS}(\eta,\phi) \propto -\arctan\left(\sqrt{\frac{1-\eta}{\eta}}\right)(e^{-i\phi}\hat{a}_1\hat{a}^\dagger_2 - e^{+i\phi}\hat{a}^\dagger_1\hat{a}_2)$$

which couples the optical modes $\hat{a}_1$ and $\hat{a}_2$. The sum of the third and fourth terms in the interaction Hamiltonian components adopt a similar form to that of the beam-splitter Hamiltonian.

$$\hat{H}^{(\eta\nu)}_{BS,nm} \propto \hbar\Omega^{(\eta\nu)}(\mathbf{k},\mathbf{k}')\left[\hat{a}_\eta(\mathbf{k})\hat{a}^\dagger_\nu(\mathbf{k}')e^{-i(\omega-\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)+\hat{a}^\dagger_\eta(\mathbf{k})\hat{a}_\nu(\mathbf{k}')e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)\right] \tag{24}$$

where $\eta, \nu \in \{s, p\}$. One inconsistency however is that the beam-splitter Hamiltonian involves the *subtraction* of terms while our interaction Hamiltonian involves the *addition* of these terms. [Nico] **Whether or not this changes the fundamental physics is unclear to me. I've seen both versions with a plus or a minus in the literature.**

4.3 Two-Mode Squeezing from Interaction Hamiltonian

The single-mode and two-mode squeezing Hamiltonians are given by

$$\hat{H}_{1s}(z) \propto \frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{\dagger 2}\right) \tag{25a}$$

$$\hat{H}_{2s}(z) \propto \frac{1}{2}\left(z^*\hat{a}_1\hat{a}_2 - z\hat{a}^\dagger_1\hat{a}^\dagger_2\right) \tag{25b}$$

where $z \in \mathbb{C}$ is a dimensionless squeezing parameter while $\hat{a}_1, \hat{a}^\dagger_1$ and $\hat{a}_2, \hat{a}^\dagger_2$ are the ladder operators for two different optical modes. The addition of the first and second terms in our interaction Hamiltonian components has a form comparable to the two-mode squeezing Hamiltonian,

$$\hat{H}^{(\eta\nu)}_{2S,nm} \propto \hbar\Omega^{(\eta\nu)}(\mathbf{k},\mathbf{k}')\left[\hat{a}_\eta(\mathbf{k})\hat{a}_\nu(\mathbf{k}')e^{-i(\omega-\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)+\hat{a}^\dagger_\eta(\mathbf{k})\hat{a}^\dagger_\nu(\mathbf{k}')e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)\right] \tag{26}$$

such that we can immediately identify the squeezing parameter as $z_{nm} \propto \Omega^{(\eta\nu)}(\mathbf{k},\mathbf{k}')\tilde{\psi}_{nm}(\mathbf{k}_\parallel+\mathbf{k}'_\parallel)e^{-i(\omega+\omega')t}$ with appropriate polarization index on the effective frequency. Note that there is a unique squeezing parameter for each mechanical mode of the pliable boundary. Moreover, the phase of the squeezing parameter is time-dependent. If $\mathbf{k} \neq \mathbf{k}'$ we recover the 2-mode squeezing Hamiltonian while if $\mathbf{k} = \mathbf{k}'$ we recover the the 1-mode squeezing Hamiltonian. As in the case of the beam-splitter Hamiltonian, our expression involves the addition of terms instead of the subtraction of terms.

5 Interaction Modes for an Arbitrary 2D Optomechanical Resonator

The annihilation operator for any mode can be defined in terms of its spectral decomposition in the plane wave operators

$$\hat{a} = \sum_{j=s,p}\int_K d\mathbf{k}\tilde{\phi}_j(\mathbf{k})\hat{a}_j(\mathbf{k}) \tag{27}$$

where the mode is square normalized,

$$\sum_{j=s,p}\int_K d\mathbf{k}|\tilde{\phi}_j(\mathbf{k})|^2 = 1 \tag{28}$$

Equation 28 comes from imposing the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and is derived in appdendix C. In this section we will derive the interaction modes for a strong driving input mode with operator $\hat{a}_0$. Our procedure for doing so will involve four approximations

1. Linearize the input mode operator (strong drive field approximation)

2. Evaluate the interaction Hamiltonian and drop terms of second order in the optical field operators (ignore interaction energies coming from exchanges with the vacuum)
3. Enforce frequency coupling (output mode frequency should be near the input mode frequency)
4. Apply rotating wave approximation (drop doubly oscillating terms in the squeezing hamiltonian)

We assume the input mode is characterized by a strong driving field such that mode operator can be decomposed into a classical and quantum component.

$$\hat{a}_0 \to a_0 + \hat{a}_0 = \sum_{j=s,p} \int_K d\mathbf{k}\tilde{\phi}_{0j}(\mathbf{k})\Big[a_0\tilde{\phi}^*_{0j}(\mathbf{k}) + \hat{a}_j(\mathbf{k})\Big] \tag{29}$$

By inspection, we see that each plane wave operator gets linearized as,

$$\hat{a}_j(\mathbf{k}) \to a_0\tilde{\phi}^*_{0j}(\mathbf{k}) + \hat{a}_j(\mathbf{k}). \tag{30}$$

Inserting the linearized plane wave operators into the interaction Hamiltonian terms we have,

$$\begin{aligned}\hat{H}^{(\eta\nu)}_{nm} = \hat{c}_{nm}(t)\left(\frac{\hbar}{8\pi^2}\right)\int_K d\mathbf{k}\int_{K'} d\mathbf{k}'\Omega^{(\eta\nu)}(\mathbf{k},\mathbf{k}')\Big\{&\Big[a_0\tilde{\phi}^*_{0\eta}(\mathbf{k}) + \hat{a}_\eta(\mathbf{k})\Big]\Big[a_0\tilde{\phi}^*_{0\nu}(\mathbf{k}') + \hat{a}_\nu(\mathbf{k}')\Big]e^{-i(\omega+\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) + H.C.\\ &+\Big[a_0\tilde{\phi}^*_{0\eta}(\mathbf{k}) + \hat{a}_\eta(\mathbf{k})\Big]\Big[a_0^*\tilde{\phi}_{0\nu}(\mathbf{k}') + \hat{a}^\dagger_\nu(\mathbf{k}')\Big]e^{-i(\omega-\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\\ &+\Big[a_0^*\tilde{\phi}_{0\eta}(\mathbf{k}) + \hat{a}^\dagger_\eta(\mathbf{k})\Big]\Big[a_0\tilde{\phi}^*_{0\nu}(\mathbf{k}') + \hat{a}_\nu(\mathbf{k}')\Big]e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\Big\}\end{aligned} \tag{31}$$

Expanding the squares and ignoring the terms of second order in the field operators we have,

$$\begin{aligned}\hat{H}^{(\eta\nu)}_{int,,nm} = \hat{c}_{nm}(t)\left(\frac{\hbar}{8\pi^2}\right)&\int_K d\mathbf{k}\int_{K'} d\mathbf{k}'\Omega^{(\eta\nu)}(\mathbf{k},\mathbf{k}')\\ &\Big\{\Big[a_0^2\tilde{\phi}^*_{0\eta}(\mathbf{k})\tilde{\phi}^*_{0\nu}(\mathbf{k}') + a_0\tilde{\phi}^*_{0\eta}(\mathbf{k})\hat{a}_\nu(\mathbf{k}') + a_0\tilde{\phi}^*_{0\nu}(\mathbf{k}')\hat{a}_\eta(\mathbf{k}) + \mathcal{O}(\hat{a}^2)\Big]e^{-i(\omega+\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel + \mathbf{k}'_\parallel) + H.C.\\ &+\Big[|a_0|^2\tilde{\phi}^*_{0\eta}(\mathbf{k})\tilde{\phi}_{0\nu}(\mathbf{k}') + a_0\tilde{\phi}^*_{0\eta}(\mathbf{k})\hat{a}^\dagger_\nu(\mathbf{k}') + a_0^*\tilde{\phi}_{0\nu}(\mathbf{k}')\hat{a}_\eta(\mathbf{k}) + \mathcal{O}(\hat{a}^2)\Big]e^{-i(\omega-\omega')t}\tilde{\psi}^*_{nm}(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\\ &+\Big[|a_0|^2\tilde{\phi}_{0\eta}(\mathbf{k})\tilde{\phi}^*_{0\nu}(\mathbf{k}') + a_0^*\tilde{\phi}_{0\eta}(\mathbf{k})\hat{a}_\nu(\mathbf{k}') + a_0\tilde{\phi}^*_{0\nu}(\mathbf{k}')\hat{a}^\dagger_\eta(\mathbf{k}) + \mathcal{O}(\hat{a}^2)\Big]e^{i(\omega-\omega')t}\tilde{\psi}_{nm}(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\Big\}\end{aligned} \tag{32}$$

The standard practice here is actually to shift into the interaction picture by transforming the entire Hamiltonian by

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_I = e^{i\hat{H}_0 t/\hbar}\hat{H}e^{-i\hat{H}_0 t/\hbar}$$

$$\implies \hat{H}_{0,I} = \hat{H}_0$$

$$\implies \hat{H}_{1,I} = e^{i\hat{H}_0 t/\hbar}\hat{H}_1 e^{-i\hat{H}_0 t/\hbar}$$

which is done in the Jaynes-Cummings model, for instance. **See OPTI 544 Notes**

Equation 32 is the linearized interaction Hamiltonian. As it stands, it is not particularly enlightening. So, next we will make three simplifying assumptions. First we assume that the input field is monochromatic. Thus the input mode functions become $\tilde{\phi}_{0j}(\mathbf{k}) = \delta(k - k_0)\tilde{\phi}_{0j}(\theta,\phi)$ in spherical coordinates over k-space. The mode functions $\tilde{\phi}_{0j}$ now apply weights to all k-vectors on a sphere of radius k_0. The spatial variation of the mode is encoded in the weight applied to k-vectors propagating at different angles. The second assumption we make is that plane waves near the drive frequency ω_0 are the only ones which couple strongly to the incident drive field. Nothing in the math suggests that this is immediately valid, however it stresses our physical intuition to believe that the scattered optical modes have wildly different frequency from the incident light (at least for strongly coupled modes). Hence we artificially introduce an frequency coupling factor $f(\omega,\omega') = \delta(\omega - \omega_0)\delta(\omega' - \omega_0)$ in the integrand. **[Nico] Given the group's discussion on 10/18/23 it seems that the scattered mode is typically of a different frequency than the incident mode. This is at least true when we consider optical cavities since the displacement of the end mirror changes the cavity frequency. In the case of a one-sided boundary though its unclear to me whether the same effect is present. Shifting a**

one-sided mirror would simply change the phase of each plane wave mode but not the mode frequencies themselves. The third assumption follows from the second in that we take the rotating wave approximation and drop terms oscillating at frequency $2\omega_0$. **[Nico] I take the RWA for no good reason other than that its typically done in the literature)**. With these simplifications, our interaction Hamiltonian terms take the form,

$$\hat{H}_{nm}^{(\eta\nu)} \approx \hat{c}_{nm}(t)\left(\frac{\hbar\omega_0}{8\pi^2}\right)\int_{K|k_0} d\mathbf{k}\int_{K'|k_0} d\mathbf{k}' g^{(\eta\nu)}(\mathbf{k},\mathbf{k}') \left\{\left[|a_0|^2\tilde{\phi}_{0\eta}^*(\mathbf{k})\tilde{\phi}_{0\nu}(\mathbf{k}') + a_0\tilde{\phi}_{0\eta}^*(\mathbf{k})\hat{a}_\nu^\dagger(\mathbf{k}') + a_0^*\tilde{\phi}_{0\nu}(\mathbf{k}')\hat{a}_\eta(\mathbf{k})\right]\tilde{\psi}_{nm}^*(\mathbf{k}_\parallel - \mathbf{k}'_\parallel) + H.C.\right\} \tag{33}$$

where the domain of integration $K|k_0$ is the surface of a sphere with radius k_0. From here, the interaction modes can immediately be identified as those terms of first order in the plane-wave field operators. Using the fact that the mechanical modes are real functions such that, $\tilde{\psi}_{nm}^*(-\mathbf{k}_\parallel) = \tilde{\psi}_{nm}(\mathbf{k}_\parallel)$, we find two interaction modes.

$$\hat{a}_{nm}^{(s)} \propto \sum_{j=s,p}\int_{K|k_0} d\mathbf{k}\left[\int_{K'|k_0} d\mathbf{k}' g^{(sj)}(\mathbf{k}',\mathbf{k})\tilde{\phi}_{0j}(\mathbf{k}')\tilde{\psi}_{nm}(\mathbf{k}'_\parallel - \mathbf{k}_\parallel)\right]\hat{a}_j(\mathbf{k}) \tag{34a}$$

$$\hat{a}_{nm}^{(p)} \propto \sum_{j=s,p}\int_{K|k_0} d\mathbf{k}\left[\int_{K'|k_0} d\mathbf{k}' g^{(pj)}(\mathbf{k}',\mathbf{k})\tilde{\phi}_{0j}(\mathbf{k}')\tilde{\psi}_{nm}(\mathbf{k}'_\parallel - \mathbf{k}_\parallel)\right]\hat{a}_j(\mathbf{k}) \tag{34b}$$

where $g^{(ps)} = g^{(sp)}$ has been introduced for notational symmetry. We see that equations 34 have identical form to 27 where the weighting functions to the $\hat{a}_j(\mathbf{k})$ operators are the plane wave representation of interaction modes themselves. That is, we can write the interaction modes as,

$$\phi_{nm,j}^{(\eta)}(\mathbf{k}|k_0) = \frac{1}{\sqrt{N_{nm}^{(\eta)}}}\int_{K'|k_0} d\mathbf{k}' g^{(\eta j)}(\mathbf{k}',\mathbf{k})\tilde{\phi}_{0j}(\mathbf{k}')\tilde{\psi}_{nm}(\mathbf{k}'_\parallel - \mathbf{k}_\parallel) \tag{35a}$$

where $\eta, j \in \{s,p\}$ and $N_{nm}^{(\eta)}$ is a normalization factor with units of [area^{-1}]. Re-introducing the operator $\hat{a}_0$ corresponding to the input field, we have the following interaction Hamiltonian up to an additive constant.

$$\hat{H}_{nm} = \hat{c}_{nm}(t)\left(\frac{\hbar\omega_0}{8\pi^2}\right)\left\{\sqrt{N_{nm}^{(s)}}\left[\hat{a}_0^\dagger\hat{a}_{nm}^{(s)} + \hat{a}_{nm}^{(s)\dagger}\hat{a}_0\right] + \sqrt{N_{nm}^{(p)}}\left[\hat{a}_0^\dagger\hat{a}_{nm}^{(p)} + \hat{a}_{nm}^{(p)\dagger}\hat{a}_0\right]\right\} \tag{36}$$

We can see in the equation above that the normalization factor is actually related to the coupling coefficient between the input mode and the interaction mode. **[Nico] In fact, N_{nm} turns out to be the same as Jack's β factor as we'll come to find in the next sections.**

5.1 Paraxial Approximation

In the paraxial approximation we assume that the incoming field can be decomposed into a sum of plane waves, each with near normal-incidence. That is, $|\tilde{\phi}_{0j}(\theta,\phi)|^2$ is concentrated around $\theta \approx 0$ and tapers off sufficiently quickly at higher angles of incidence. Under this approximation, we can say that $k_z \approx k_0$ for all constituent vectors of the incoming field. This is visually demonstrated in figure 2. Applying this approximation to the geometric factors we have,

$$g^{(ss)}(\mathbf{k},\mathbf{k}') \to g^{(ss)}(\mathbf{k}_\parallel,\mathbf{k}'_\parallel) = (\bar{\mathbf{k}}_\parallel \cdot \bar{\mathbf{k}}'_\parallel) = \cos(\phi - \phi') \tag{37a}$$

$$g^{(sp)}(\mathbf{k},\mathbf{k}') \to g^{(sp)}(\mathbf{k}_\parallel,\mathbf{k}'_\parallel) = (\bar{\mathbf{k}}_\parallel \times \bar{\mathbf{k}}'_\parallel)\cdot(-\bar{\mathbf{z}}) = \sin(\phi - \phi') \tag{37b}$$

$$g^{(pp)}(\mathbf{k},\mathbf{k}') \to g^{(pp)}(\mathbf{k}_\parallel,\mathbf{k}'_\parallel) = (\bar{\mathbf{k}}_\parallel \cdot \bar{\mathbf{k}}'_\parallel) = \cos(\phi - \phi') \tag{37c}$$

Under these approximations the two interaction modes become degenerate $\hat{a}_{nm}^{(s)} = \hat{a}_{nm}^{(p)} \equiv \hat{a}_{nm}$. **Nico: At this point I ignore the geometric factors all together. Intuitively they should have no impact in the paraxial regime. At normal incidence, the orthogonal polarizations are both parallel to the surface. Doing this also recovers the classical result.** Ignoring the geometric terms, we find the interaction mode operator in the paraxial regime to be

$$\hat{a}_{nm} = \frac{1}{\sqrt{N_{nm}}}\sum_{j=s,p}\int_{\mathbb{R}^2} d\mathbf{k}_\parallel\left[\int_{\mathbb{R}^2} d\mathbf{k}'_\parallel\tilde{\phi}_{0j}(\mathbf{k}')\tilde{\psi}_{nm}(\mathbf{k}'_\parallel - \mathbf{k}_\parallel)\right]\hat{a}_j(\mathbf{k}_\parallel) \tag{38}$$

We see that the interaction modes themselves correspond to a 2D convolution between the Fourier transforms of the input mode and the mechanical membrane mode.

$$\tilde{\phi}_{nm,j}(\mathbf{k}_\parallel) = \frac{1}{\sqrt{N_{nm}}}(\tilde{\phi}_{0j} \star\star \tilde{\psi}_{nm})(\mathbf{k}_\parallel) \tag{39}$$

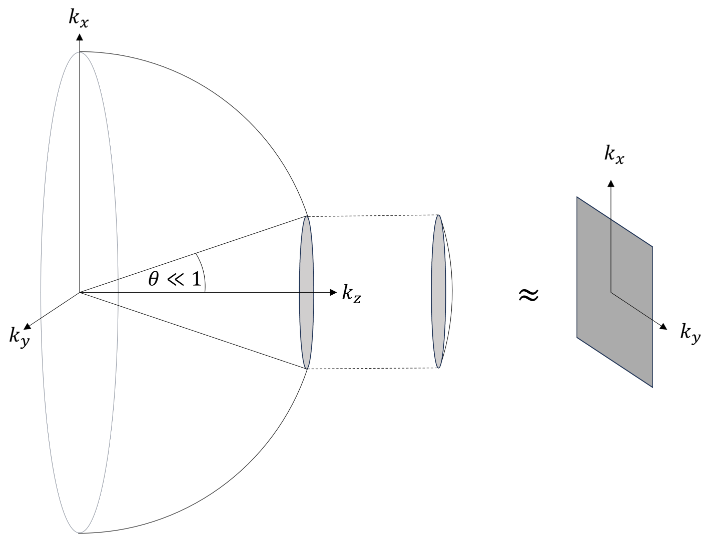

Figure 2: A visual illustration of the paraxial approximation. If the k-space mode functions of a monochromatic input field are concentrated around the z-axis then the field can be well approximated by a 2D function over the cartesian coordinates $\mathbf{k}_{\|}$.

Therefore, by the convolution theorem, we see that the transverse spatial profile of the interaction mode is simply the product of the input mode profile and the mechanical mode profile. This agrees with the classical result.

$$\phi_{nm,j}(\mathbf{r}_{\|}) = \frac{1}{\sqrt{N_{nm}}}\phi_{0j}(\mathbf{r}_{\|})\psi_{nm}(\mathbf{r}_{\|}) \tag{40}$$

At this point is valuable to write down the normalization factor explicitly.

$$N_{nm} = \sum_{j=s,p}\int_{\mathbb{R}^2} d\mathbf{r}_{\|}\left(\phi_{0j}(\mathbf{r}_{\|})\psi_{nm}(\mathbf{r}_{\|})\right)^*\left(\phi_{0j}(\mathbf{r}_{\|})\psi_{nm}(\mathbf{r}_{\|})\right) \tag{41a}$$

$$= \sum_{j=s,p}\int_{\mathbb{R}^2} d\mathbf{r}_{\|}|\phi_{0j}(\mathbf{r}_{\|})|^2|\psi_{nm}(\mathbf{r}_{\|})|^2 = \beta_{nm} \tag{41b}$$

We see from equation 36 that the square root of the normalization factor determines the coupling strength between modes. In particular, the coupling strength is determined by how much the intensity profile of the drive field (square of the mode profile) overlaps with the square of the membrane mode. **[Nico] This normalization factor is identical to the beta factor in Jack's note. To avoid confusion, here I've define ϕ as the input field *amplitude* whereas Jack defines ϕ as the input field *intensity* profile. Thus I have $|\phi|^2$ where Jack has ϕ.**

5.2 Gram-Schmidt Orthogonalization of the Interaction Mode Bosonic Operator

Next we will deconstruct the interaction mode bosonic operator as a linear combination of commuting bosonic operators $[\hat{a}_0, \hat{a}_\perp^\dagger] = 0$. This is effectively performing a gram-schmidt orthogonalization of the interaction mode.

$$\hat{a}_{nm} = u\hat{a}_0 + v\hat{a}_\perp \tag{42}$$

Our goal is to determine the weights u, v. By requirement, we have that $[\hat{a}_{nm}, \hat{a}_{nm}^\dagger] = 1$ which implies that $|u|^2+|v|^2 = 1$. We also briefly note a convenient property of bosonic operators: The commutator of two field operators is equal to the inner product between their underlying modes $[\hat{a}_1, \hat{a}_2^\dagger] = \langle\phi_2, \phi_1\rangle$ (see proof in appendix D). Therefore, the bosonic operators $\hat{a}_0$ and $\hat{a}_{nm}^\dagger$ commute iff the interaction mode is orthogonal to the input mode. One obvious example is if the driving input mode is an even function (e.g. gaussian beam) and the membrane mode is an odd function (i.e. antisymmetric displacement profile). Then the interaction mode is necessarily odd and its inner product with the input mode is zero. Thus, by evaluating commutators, we can determine the coeffecients directly,

$$[\hat{a}_0, \hat{a}_{nm}^\dagger] = u^* = \sum_{j=s,p}\langle\phi_{nm,j}, \phi_{0j}\rangle \tag{43}$$

$$[\hat{a}_\perp, \hat{a}_{nm}^\dagger] = v^* = \sum_{j=s,p}\langle\phi_{nm,j}, \phi_{\perp j}\rangle \tag{44}$$

The underlying mode for $\hat{a}_\perp$ is simply the Gram-Schmidt orthogonalization of $\phi_{nm,j}$ where we have removed the projection onto $\phi_0 j$;

$$\phi_{\perp j} = \frac{\phi_{nm,j} - \langle \phi_{0j}, \phi_{nm,j} \rangle \phi_{0j}}{\sqrt{\langle \phi_{nm,j} - \langle \phi_{0j}, \phi_{nm,j} \rangle \phi_{0j} \rangle}} \tag{45}$$

In equation 45 we have used $\langle f \rangle \equiv \langle f, f \rangle$ to denote the inner product of a function with itself.

Aside: There is something curious about equation 42. It implies that a single photon state of one mode can be written also as a superposition of single photon states in a collection of other modes (with non-zero projections onto the former). Here is the idea (neglecting the polarization indices). Suppose we act the interaction mode raising operator on the vacuum state.

$$\hat{a}^\dagger_{nm} |0\rangle = |1\rangle_{\phi_{nm}}$$

We get a single photon in the interaction mode, indicated by the subscript, and vacuum in all the remaining modes orthogonal to the interaction mode (we could indicate this by a trailing $|0\rangle$ but this complicates the notation). As we have defined things, this is equivalent to

$$\hat{a}^\dagger_{nm} |0\rangle = (u^* \hat{a}^\dagger_0 + v^* \hat{a}^\dagger_\perp) |0\rangle = u^* |1\rangle_{\phi_0} + v^* |1\rangle_{\phi_\perp}$$

This is rather curious as we have just shown that a single photon state in one mode can also be thought of as a super-position of single photon states in multiple other modes. That is,

$$|1\rangle_{\phi_{nm}} = u^* |1\rangle_{\phi_0} + v^* |1\rangle_{\phi_\perp}$$

What if we put in two photons in the interaction mode?

$$\frac{1}{\sqrt{2}} \left(\hat{a}^\dagger_{nm}\right)^2 |0\rangle = |2\rangle_{\phi_{nm}} = (u^*)^2 |2\rangle_{\phi_0} + 2u^* v^* |1\rangle_{\phi_0} |1\rangle_{\phi_\perp} + (v^*)^2 |2\rangle_{\phi_\perp}$$

We see that the same 2-photon state represented in a different (orthogonal) mode set becomes a superposition of all possible ways the two photons could be distributed into either mode. Note that this is actually just like a beam-splitter description.

$$\begin{bmatrix} \hat{a}_{nm} \\ \hat{a}_{arb} \end{bmatrix} = \begin{bmatrix} u & v \\ \sim & \sim \end{bmatrix} \begin{bmatrix} \hat{a}_0 \\ \hat{a}_\perp \end{bmatrix}$$

However, instead of imposing any active action on the field we've simply made a passive transformation on the underlying modal description.

5.3 Reduced Hamiltonian

In the rotating frame of a strong monocrhomatic drive laser with field operator $\hat{a}_0$ corresponding to paraxial mode $\phi_{0j}(\mathbf{r}_\parallel)$ the total Hamiltonian reduces to,

$$\boxed{\hat{H}_{tot} = \hbar\omega_0 \left(\hat{a}^\dagger_0 \hat{a}_0 + \frac{1}{2}\right) + \sum_{nm} \hbar\omega_{nm} \left(\hat{b}^\dagger_{nm} \hat{b}_{nm} + \frac{1}{2}\right) + \hbar\omega_0 \sum_{nm} \frac{\sqrt{N_{nm}}}{4\pi^2} (\hat{b}_{nm} e^{-i\omega_{nm} t} + \hat{b}^\dagger_{nm} e^{i\omega_{nm} t}) \left[\hat{a}^\dagger_0 \hat{a}_{nm} + \hat{a}^\dagger_{nm} \hat{a}_0\right]} \tag{46}$$

The first term is simply the electromagnetic energy in the mode corresponding to the input drive field. The second term is the energy of the mechanical oscillator over all of its modes. The third term describes the interaction energy between the input mode and the scattered mode mediated by oscillations in the mechanical membrane.

5.4 Time Evolution of a Pure State

The Hamiltonian in 46 is time-dependent. Fortunately, the Hamiltonian commutes with itself at all times $[\hat{H}_{tot}(t), \hat{H}_{tot}(t')] = 0 \quad \forall t, t'$ and thus lends itself nicely to an investigation of time-evolution for the state.

5.5 Zero-Point Fluctuation for Each Mechanical Mode

The zero point fulctuation

$$X^{ZPF}_{nm} = \sqrt{\frac{\hbar}{2 m_{nm} \omega_{nm}}}$$

where the effective mass of the nm^{th} mechanical mode is

$$m_{nm} = \sigma_u \iint |\psi_{nm}(x, y)|^2 dx dy$$

and σ_u is the surface density of the membrane with units of mass per unit area. It may appear as though the integral over the square of the membrane mode is 1. However, in reality we define ψ_{nm} to be max-normalized to 1 (i.e. $max_{x,y} \psi_{nm}(x, y) = 1$) so that the mode amplitude coefficients are in units of length. This normalization gives each mode a different effective mass.

5.6 Quantum Radiation Pressure

The effective radiation pressure acting on a given mechanical mode can be found by differentiating the interaction Hamiltonian with respect to the mechanical mode displacement.

$$\hat{F}_{nm} = \frac{\partial \hat{H}_{int}}{\partial \hat{c}_{nm}} = \frac{\partial \left[\hat{H}^{(ss)}_{int,nm} + \hat{H}^{(sp)}_{int,nm} + \hat{H}^{(pp)}_{int,nm} \right]}{\partial \hat{c}_{nm}}$$

5.7 Imprecision Back-Action Product

5.8 Evolution of an Input State

TODO The annihilation operator for any mode can be expressed as

$$\hat{a}_\phi = \sum_{j=s,p} \int_K d\mathbf{k} \phi_j(\mathbf{k}) \hat{a}_j(\mathbf{k})$$

where ϕ is square-normalized function over the space of plane waves and satisfies the system boundary conditions. We wish to evaluate how this operator evolves under the interaction Hamiltonian.

$$\hat{a}_\phi(t) = \hat{U}^\dagger(t, t_0) \hat{a}_\phi \hat{U}(t, t_0)$$

where in general, the time evolution operator solves the differential equation

$$i\hbar \partial_t \hat{U}(t, t_0) = \hat{H} \hat{U}(t, t_0)$$

If the Hamiltonian evaluated at different times commutes $[\hat{H}(t), \hat{H}(t')] = 0 \quad \forall \quad t \neq t'$, then the time evolution operator can simply be expressed as,

$$\hat{U}(t, t_0) = \exp\left(-\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}(t') dt' \right)$$

Otherwise, the time evolution operator is given by a 'Dyson ordering'

$$\hat{U}(t, t_0) = 1 + \sum_{n=1}^{\infty} \int_{t_0}^{t} d\tau_n \int_{t_0}^{\tau_n} d\tau_{n-1} \cdots \int_{t_0}^{\tau_2} d\tau_1 \hat{H}(\tau_n) \hat{H}(\tau_{n-1}) \cdots \hat{H}(\tau_1)$$

5.9 Heisenberg Equations of Motion for Field Operators

TODO

5.10 Interaction with a Coherent State

TODO

In this section, we try to derive the modes that interact strongly to a coherent state input via $\hat{H}_{int}$. Consider the multimode coherent state $|\alpha_j(\mathbf{k})\rangle$ where $\alpha_j(\mathbf{k})$ is a function encapsulating the 'amount' of displacement in each plane-wave mode. We may equivalently write the multimode coherent state in the plane wave basis as a single-mode coherent state of the mode $\phi_j(\mathbf{k})$ for $j = s, p$ displaced by amount α via,

$$|\alpha\rangle_\phi = |\alpha_j(\mathbf{k})\rangle = |\alpha \phi_j(\mathbf{k})\rangle$$

We may rewrite this coherent state as the multimode displacement operator acting on the vacuum state.

$$|\alpha_j(\mathbf{k})\rangle = \hat{\mathcal{D}}\big(\alpha_j(\mathbf{k})\big) |0\rangle = \exp\left[\sum_{j=s,p} \int_K d\mathbf{k} \alpha_j(\mathbf{k}) \hat{a}^\dagger_j(\mathbf{k}) - \alpha^*_j(\mathbf{k}) \hat{a}(\mathbf{k}) \right] |0\rangle$$

Since the displacement operator is unitary, one thing we could try is transforming the interaction Hamiltonian to be in the 'frame' of the incident beam. I think this is the same as considering the so-called 'interaction picture' of the Hamiltonian. Consider the transformed interaction Hamiltonian

$$\hat{H}^{'}_{int} = \Big(\hat{\mathcal{D}}\big(\alpha_j(\mathbf{k})\big) \otimes \hat{I}_b \Big) \hat{H}_{int} \Big(\hat{\mathcal{D}}\big(\alpha_j(\mathbf{k})\big) \otimes \hat{I}_b \Big)^\dagger$$

where $\hat{I}_b$ is the identity operator on the Hilbert space associated with the mechanical oscillator.

5.11 Coherent State of a Generalized Mode

TODO Coherent states enjoy the a property called the Slicing Theorem which states that a multi-mode coherent state can always be expressed as a single-mode coherent state under appropriate choice of mode. This property comes from the fact that coherent states are eigenstates of the annihilation operators. A multimode creation operator can be expressed as

$$\hat{a}_\phi^\dagger = \sum_j \int_K d\mathbf{k}\phi_j(\mathbf{k})\hat{a}_j(\mathbf{k})$$

such that the displacement operator for this mode is

$$\hat{\mathcal{D}}_\phi(\alpha) = \exp\Big(\alpha\hat{a}_\phi^\dagger - \alpha^*\hat{a}_\phi\Big)$$

5.12 Switching into Rotating Frame

TODO Supposing our incident coherent state $\alpha(\mathbf{k})$ has frequency ω_α, from Aspelmeyer [page 30], we may apply a unitary transformation $\hat{U} = \exp\big(i\omega_\alpha\hat{a}^\dagger(\alpha)\hat{a}(\alpha)t\big)$ to the Hamiltonian to rotate into the frame of the incident probe field and make the driving terms time-independent.

$$\hat{H}' = \hat{U}^\dagger\hat{H}\hat{U}$$

$$\hat{U}^\dagger(\hat{a}^\dagger e^{-i\omega_\alpha t} + \hat{a}e^{i\omega_\alpha t})\hat{U} = \hat{a}^\dagger + \hat{a}$$

6 The Square Membrane

Let us now consider an example. Suppose we allow a square region of the PEC half-space boundary surface to be pliable. Inside this region the surface is free to oscillate in its mechanical modes while outside the region the half-space boundary is assumed to be flat. We will explore the implications of the interaction Hamiltonian derived using **Method 1: Plane-Wave Expansion** for this particular membrane geometry. The Fourier Transform of the mechanical modes for a square membrane are given by,

$$\begin{aligned}
\tilde{\psi}_{nm}(k_x,k_y) &= \mathcal{F}_{k_x}\Big\{\sin\Big(\frac{n\pi}{L}(x-\frac{L}{2})\Big)\texttt{rect}(x/L)\Big\}\mathcal{F}_{k_y}\Big\{\sin\Big(\frac{m\pi}{L}(y-\frac{L}{2})\Big)\texttt{rect}(y/L)\Big\} \\
&= i\sqrt{\frac{\pi}{2}}\Big[e^{in\pi/2}\delta(k_x-\frac{n\pi}{L}) - e^{-in\pi/2}\delta(k_x+\frac{n\pi}{L})\Big] * L\,\mathrm{sinc}(k_xL/2) \\
&\quad i\sqrt{\frac{\pi}{2}}\Big[e^{im\pi/2}\delta(k_y-\frac{m\pi}{L}) - e^{-im\pi/2}\delta(k_y+\frac{m\pi}{L})\Big] * L\,\mathrm{sinc}(k_yL/2) \\
&= i\sqrt{\frac{\pi}{2}}\Big[i^n\delta(k_x-n\pi/L) - i^{-n}\delta(k_x+n\pi/L)\Big] * L\mathrm{sinc}(k_xL/2) \\
&\quad i\sqrt{\frac{\pi}{2}}\Big[i^m\delta(k_y-m\pi/L) - i^{-m}\delta(k_y+m\pi/L)\Big] * \ L\mathrm{sinc}(k_yL/2) \\
&= \frac{\pi}{2}\Big[w_{nm}^{(1)}\delta(\mathbf{k}_\parallel - \mathbf{k}_{nm\parallel}^{(1)}) + w_{nm}^{(2)}\delta(\mathbf{k}_\parallel - \mathbf{k}_{nm\parallel}^{(2)}) \\
&\qquad w_{nm}^{(3)}\delta(\mathbf{k}_\parallel - \mathbf{k}_{nm\parallel}^{(3)}) + w_{nm}^{(4)}\delta(\mathbf{k}_\parallel - \mathbf{k}_{nm\parallel}^{(4)})\Big] ** \ L^2\mathrm{sinc}(k_xL/2)\,\mathrm{sinc}(k_yL/2)
\end{aligned}$$

where in the last line we have defined, $\mathbf{k}_{nm\parallel}^{(j)}$ and $w_{nm}^{(j)}$ for $j = 1,2,3,4$ as,

$$\begin{aligned}
\mathbf{k}_{nm\parallel}^{(1)} &= \frac{\pi}{L}(+n,+m) & w_{nm}^{(1)} &= -i^{n+m} \\
\mathbf{k}_{nm\parallel}^{(2)} &= \frac{\pi}{L}(+n,-m) & w_{nm}^{(2)} &= +i^{n-m} \\
\mathbf{k}_{nm\parallel}^{(3)} &= \frac{\pi}{L}(-n,+m) & w_{nm}^{(3)} &= +i^{-n+m} \\
\mathbf{k}_{nm\parallel}^{(4)} &= \frac{\pi}{L}(-n,-m) & w_{nm}^{(4)} &= -i^{-n-m}
\end{aligned}$$

The coupling strength between plane waves is concentrated around $\mathbf{k}_\parallel \pm \mathbf{k}'_\parallel = \mathbf{k}_{\parallel nm}^{(j)}$ due to the dirac deltas convolved with a sinc envelope. These vectors $\mathbf{k}_{nm}^{(j)}$ form vertices of a rectangle (a square in the case of $n = m$) in the frequency

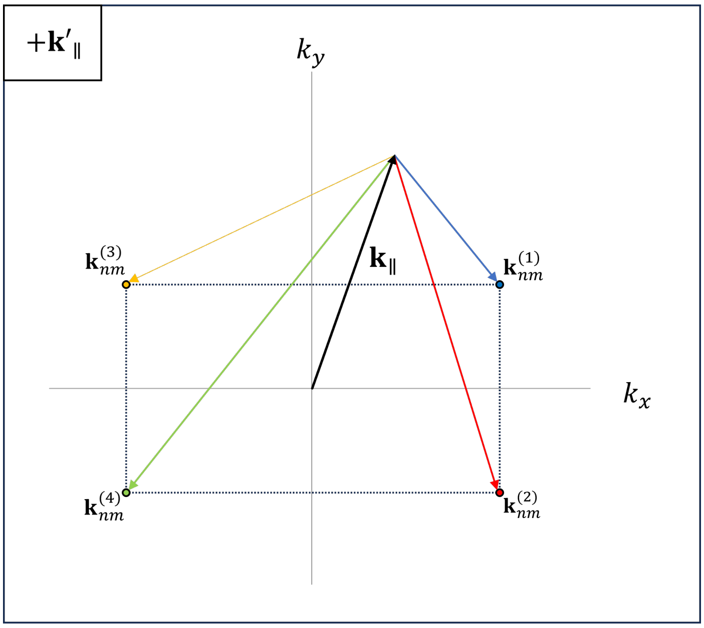

Figure 3: Colored vectors represent the transverse components of plane waves that couple most strongly with the input plane wave (black vectors). Note that for each colored vector $\mathbf{k}_{\parallel} \pm \mathbf{k}'_{\parallel} = \mathbf{k}_{nm}^{(j)}$. Thus there are two high coupling vectors $\pm\mathbf{k}'_{\parallel}$ for each rectangle vertex.

plane (k_x, k_y), offering a nice geometric picture of the dominant optical mode interactions enhanced by the presence of excitations in mechanical mode ψ_{nm} (see figure 3). The polarization terms of the interaction Hamiltonian are thus well-approximated by,

$$\begin{aligned}\hat{H}^{(\eta\nu)}_{int,nm} \approx \hat{c}_{nm}\left(\frac{\hbar}{16\pi}\right)\sum_{j=1}^{4}\int_K d\mathbf{k}\int_0^\infty dk'_z \Big[\Omega^{(\eta\nu)}(\mathbf{k},{}^{+}\mathbf{k}^{(j)'}_{nm})\Big\{\hat{a}_\eta(\mathbf{k})\hat{a}_\nu({}^{+}\mathbf{k}^{(j)'}_{nm})e^{-i(\omega+\omega')t}w^{(j)*}_{nm} + H.C.\Big\} \\ +\Omega^{(\eta\nu)}(\mathbf{k},{}^{-}\mathbf{k}^{(j)'}_{nm})\Big\{\hat{a}_\eta(\mathbf{k})\hat{a}^\dagger_\nu({}^{-}\mathbf{k}^{(j)'}_{nm})e^{-i(\omega-\omega')t}w^{(j)*}_{nm} \\ + \hat{a}^\dagger_\eta(\mathbf{k})\hat{a}_\nu({}^{-}\mathbf{k}^{(j)'}_{nm})e^{i(\omega-\omega')t}w^{(j)}_{nm}\Big\}\Big]\end{aligned} \tag{47}$$

where ${}^{\pm}\mathbf{k}^{(j)'}_{nm} = [\pm(\mathbf{k}^{(j)}_{nm\parallel} - \mathbf{k}_{\parallel}), k'_z]$.

6.1 Illumination by a Driving Plane Wave

Let us now consider what happens if we have a strong s-polarized plane wave with k-vector $\mathbf{k}_0$ incident on the boundary. Linearizing the plane wave bosonic operator $\hat{a}_s(\mathbf{k}_0) \to a(\mathbf{k}_0) + \hat{a}_s(\mathbf{k}_0)$ where $a(\mathbf{k}_0) \in \mathbb{C}$ we make the approximation that all terms of second order in the optical field operators can be ignored in the interaction Hamiltonian. Dropping the integral over $d\mathbf{k}$ gives us a differential energy density over k-space of units [Energy $\times$ Volume].

$$\begin{aligned}d\hat{H}^{(s\nu)}_{int,nm} \approx \hat{c}_{nm}\left(\frac{\hbar}{16\pi}\right)\sum_{j=1}^{4}\int_0^\infty dk'_z \Big[\Omega^{(s\nu)}(\mathbf{k}_0,{}^{+}\mathbf{k}^{(j)'}_{nm})\Big\{a(\mathbf{k}_0)\hat{a}_\nu({}^{+}\mathbf{k}^{(j)'}_{nm})e^{-i(\omega_0+\omega')}w^{(j)*}_{nm} + H.C.\Big\} \\ +\Omega^{(s\nu)}(\mathbf{k}_0,{}^{-}\mathbf{k}^{(j)'}_{nm})\Big\{a(\mathbf{k}_0)\hat{a}^\dagger_\nu({}^{-}\mathbf{k}^{(j)'}_{nm})e^{-i(\omega_0-\omega')}w^{(j)*}_{nm} \\ + a^*(\mathbf{k}_0)\hat{a}_\nu({}^{-}\mathbf{k}^{(j)'}_{nm})e^{i(\omega_0-\omega')}w^{(j)}_{nm}\Big\}\Big]\end{aligned}$$

Next we'll make another simplification and assume that plane wave modes around frequency ω_0 are the only ones which couple strongly to the drive frequency. Nothing in the math suggests that this is immediately true however it stresses our physical intuition to believe that the scattered optical modes have wildly different frequency from the incident light. This intuition may be more rigorously shown by divorcing ourselves from the PEC idealization and including a frequency-dependent coefficient of reflection. Thus we artificially inject a unitless weighting function $f(\omega') = \delta(\omega' - \omega_0)$ in the integral to impose $|{}^{\pm}\mathbf{k}^{(j)'}_{nm}| = k_0$. Taking the rotating wave approximation and dropping terms oscillating at frequency $2\omega_0$, we find,

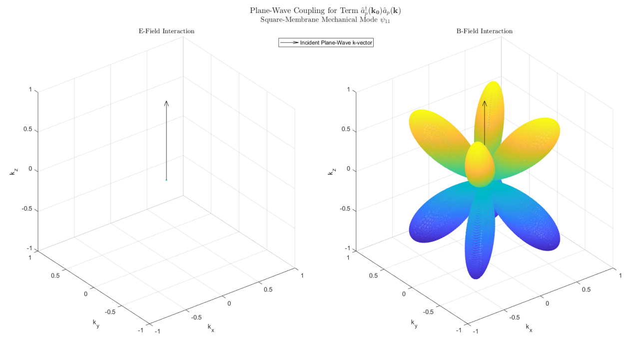

Figure 4: Coupling strength between an on-axis plane wave illuminating the ψ_{11} mechanical mode of a square membrane and all plane-waves of the same frequency as the incident wave. The coupling strength is plotted as the radial distance from the origin. Here we have included the complete description of $\tilde{\psi}_{11}$ without making any approximations. (Left) Coupling due to the electric field interaction with the mechanical mode. (Right) Coupling due to the magnetic field interaction with the mechanical mode. The interaction Hamiltonian involves only p-polarized field operators. Thus, an on-axis plane wave arriving at normal incidence at normal incidence has no involvement in optical mode coupling since the electric field component of the plane wave is necessarily s-polarized. Note that there are eight total primary lobes owed to the $\pm\mathbf{k}'_{\parallel}$ planes waves that reach the four dirac delta functions in the mechanical mode spectrum located at $\mathbf{k}^{(j)}_{nm}$.

$$d\hat{H}^{(ss)}_{int,nm} \approx \hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)\sum_{j=1}^{4}\frac{k_{0z}k'_z}{k_0^2}(\bar{\mathbf{k}}_{0\parallel}\cdot {}^{-}\bar{\mathbf{k}}^{(j)'}_{nm\parallel})\left\{a(\mathbf{k}_0)\hat{a}^\dagger_s({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)*}_{nm} + a^*(\mathbf{k}_0)\hat{a}_s({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)}_{nm}\right\} \tag{48}$$

$$d\hat{H}^{(sp)}_{int,nm} \approx -\hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)\sum_{j=1}^{4}\frac{k'_z}{k_0}(\bar{\mathbf{k}}_{0\parallel}\times {}^{-}\bar{\mathbf{k}}^{(j)'}_{nm\parallel})\cdot\bar{\mathbf{z}}\left\{a(\mathbf{k}_0)\hat{a}^\dagger_p({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)*}_{nm} + a^*(\mathbf{k}_0)\hat{a}_p({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)}_{nm}\right\} \tag{49}$$

$$d\hat{H}^{(pp)}_{int,nm} \approx 0 \tag{50}$$

At this point $d\hat{H}_{int}$ still has units of [Energy $\times$ Volume] as the integral over dk'_z was implicitly carried out. The constraint in the magnitude of the scattered k-vector in turn constrains $k'_z = \sqrt{k_0^2 - |\mathbf{k}^{(j)}_{\parallel nm} - \mathbf{k}_{0\parallel}|^2}$. For an optical drive field the magnitude of the parallel component of the k-vector is on the order of $k_{0\parallel} \sim 10^7\sin(\theta)\ [m^{-1}]$. If we are illuminating a lower order mechanical mode $n, m < 10$ of an optomechanical resonator with side length $L = 1\,mm$ then spatial frequency of the mechanical mode is on the order of $k_{\parallel nm} \sim 10^3\ [m^{-1}]$. For the parallel k-vector components to be of comparable magnitude, the angle of incidence for the plane wave must be on the order of $\theta \approx 10^{-4}\ [rads]$ which for all intents and purposes is a normally incident wave.

- For a paraxial driving wave (angle of incidence $\theta \leq 10^{-4}$) we have ${}^{-}\bar{\mathbf{k}}^{(j)'}_{nm\parallel} \approx -\mathbf{k}_{nm\parallel}$ and $k'_z \approx \sqrt{k_0^2 - k^2_{\parallel nm}} \approx k_0(1 - k^2_{\parallel nm}/k_0^2)$ where the last Taylor expansion is justified by the k-vector of the mechanical mode being many orders of magnitude smaller than the k-vector of the drive field at optical frequencies.
- For a non-paraxial driving wave we have ${}^{-}\bar{\mathbf{k}}^{(j)'}_{nm\parallel} \approx \mathbf{k}_{0\parallel}$ and $k'_z \approx \sqrt{k_0^2 - k^2_{0\parallel}} = k_{0z}$

Taking the paraxial case, we have

$$d\hat{H}^{(ss)}_{int,nm} \approx -\hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)\sum_{j=1}^{4}\cos(\theta)(1 - k^2_{\parallel nm}/k_0^2)(\bar{\mathbf{k}}_{0\parallel}\cdot\bar{\mathbf{k}}^{(j)}_{\parallel nm})\left\{a(\mathbf{k}_0)\hat{a}^\dagger_s({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)*}_{nm} + a^*(\mathbf{k}_0)\hat{a}_s({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)}_{nm}\right\} \tag{51}$$

$$d\hat{H}^{(sp)}_{int,nm} \approx \hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)\sum_{j=1}^{4}(1 - k^2_{\parallel nm}/k_0^2)(\bar{\mathbf{k}}_{0\parallel}\times\bar{\mathbf{k}}^{(j)}_{\parallel nm})\cdot\bar{\mathbf{z}}\left\{a(\mathbf{k}_0)\hat{a}^\dagger_p({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)*}_{nm} + a^*(\mathbf{k}_0)\hat{a}_p({}^{-}\mathbf{k}^{(j)'}_{nm})w^{(j)}_{nm}\right\} \tag{52}$$

These differential energies provide the interaction induced by a single s-polarized driving wave with k-vector $\mathbf{k}_0$ and strength $a(\mathbf{k}_0)$.

6.2 Interaction Modes for a Monochromatic Paraxial Mode

We can now identify what the interaction Hamiltonian is for an arbitrary spatial mode composed by a weighted sum of monochromatic s-polarized plane waves. That is, we may define a mode of frequency ω_0 with spectrum $a(\mathbf{k}_0) \equiv a_0\phi_0(\theta, \phi)$. We have effectively constrained the spectrum of this mode to exist in on the hemisphere of radius k_0 (hemisphere because we've already separated forward and backward propagating waves in the plane-wave decomposition of the Hamiltonian). If $|\phi_0(\theta,\phi)|^2$ tapers off sufficiently quickly at higher angles to satisfy the paraxial approximation, we have

$$\hat{H}^{(ss)}_{int,nm} \approx -\hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)(1-k^2_{\parallel nm}/k_0^2)\sum_{j=1}^{4} w^{(j)}_{nm} a_0^* \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\ \phi_0^*(\theta,\phi)\sin(\theta)\cos(\theta)\cos\left(\phi-\phi^{(j)}_{nm}\right)\hat{a}_s(^{-}\mathbf{k}^{(j)'}_{nm}) + H.C. \tag{53a}$$

$$\hat{H}^{(sp)}_{int,nm} \approx +\hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)(1-k^2_{\parallel nm}/k_0^2)\sum_{j=1}^{4} w^{(j)}_{nm} a_0^* \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\ \phi_0^*(\theta,\phi)\sin(\theta)\sin\left(\phi-\phi^{(j)}_{nm}\right)\hat{a}_p(^{-}\mathbf{k}^{(j)'}_{nm}) + H.C. \tag{53b}$$

Looking at equation 53 we immediately see a prescription for the interaction modes with an arbitrary monochromatic s-polarized mode $\hat{a}_0$ described by the paraxial mode-shape function $g(\theta,\phi)$.

$$\hat{a}_0 = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin(\theta)\ \phi_0(\theta,\phi)\hat{a}_s(\mathbf{k}|k_0)$$

In particular, we have an s-polarized and p-polarized interaction mode.

$$\hat{a}^{(ss)(j)}_{nm} = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin(\theta)\ \phi_0^*(\theta,\phi)\cos(\theta)\cos\left(\phi-\phi^{(j)}_{nm}\right)\hat{a}_s(^{-}\mathbf{k}^{(j)'}_{nm}|k_0) \tag{54}$$

$$\hat{a}^{(sp)(j)}_{nm} = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin(\theta)\ \phi_0^*(\theta,\phi)\sin\left(\phi-\phi^{(j)}_{nm}\right)\hat{a}_p(^{-}\mathbf{k}^{(j)'}_{nm}|k_0) \tag{55}$$

These allow us to write the interaction Hamiltonians effectively as a finite sum of two-mode couplings.

$$\hat{H}^{(s\nu)}_{int,nm} \approx \pm_\nu\ \hat{c}_{nm}\left(\frac{\hbar\omega_0}{16\pi}\right)(1-k^2_{\parallel nm}/k_0^2)\sum_{j=1}^{4} w^{(j)}_{nm}\hat{a}_0^\dagger \hat{a}^{(s\nu)(j)}_{nm} + w^{(j)*}_{nm}\hat{a}_0\hat{a}^{\dagger(s\nu)(j)}_{nm} \tag{56}$$

where $\nu \in \{s,p\}$ as before and $+$ is activated for $\nu = p$ and $-$ is activated for $\nu = s$. In principle, we could run the same derivation for a p-polarized incident wave to determine the interaction modes for an arbitrary monochromatic drive of either polarization.

A Expansion of Perturbed Integration Bounds

Consider an well-defined integral of the form shown below where the perturbation to the bound of integration is small $\delta << 1$. The

$$\int_a^{b+\delta} f(x)dx = \int_a^b f(x)dx + \int_b^{b+\delta} f(x)dx$$
$$= \int_a^b f(x)dx + F(b+\delta) - F(b)$$

Expanding the function F to first order in δ

$$F(b+\delta) \approx F(b) + \frac{dF}{dx}\,|_{x=b}\,\delta = F(b) + f(b)\delta$$

To an increasingly good approximation for small δ, we find

$$\implies \int_a^{b+\delta} f(x)dx \approx \int_a^b f(x)dx + f(b)\delta$$

B Proof of Commuting S and P Operators

The s and p electric and magnetic field operators are effectively summations over $\hat{a}_s(\mathbf{k})$ and $\hat{a}_p(\mathbf{k})$ operators where $[\hat{a}_s(\mathbf{k}), \hat{a}_p(\mathbf{k}')] = 0$. Consider then the operators

$$\hat{A} = \sum_{k\in\mathcal{A}} \hat{a}_k$$
$$\hat{B} = \sum_{k\in\mathcal{B}} \hat{b}_k$$

where $[\hat{a}_k, \hat{b}_{k'}] = 0 \quad \forall k, k' \in \mathcal{A}, \mathcal{B}$ and the sum is semantically meant to represent the addition of all elements in the sets $\mathcal{A}$ or $\mathcal{B}$. We wish to determine whether $\hat{A}$ and $\hat{B}$ commute. Writing the commutator and expanding out a single element WLOG from either sum, we find,

$$[\hat{A}, \hat{B}] = [\hat{a}_\kappa + \sum_{k \in \mathcal{A}/\kappa} \hat{a}_k, \hat{b}_{\kappa'} + \sum_{k' \in \mathcal{B}/\kappa'} \hat{b}_{k'}]$$

Now we use the commutator identity that, $[\hat{A} + \hat{B}, \hat{C} + \hat{D}] = [\hat{A}, \hat{C}] + [\hat{A}, \hat{D}] + [\hat{B}, \hat{C}] + [\hat{B}, \hat{D}]$

$$\begin{aligned}
[\hat{A}, \hat{B}] &= [\hat{a}_\kappa, \hat{b}_{\kappa'}] + [\hat{a}_\kappa, \sum_{k' \in \mathcal{B}/\kappa'} \hat{b}_{k'}] + [\sum_{k \in \mathcal{A}/\kappa} \hat{a}_k, \hat{b}'_\kappa] + [\sum_{k \in \mathcal{A}/\kappa} \hat{a}_k, \sum_{k' \in \mathcal{B}/\kappa'} \hat{b}_{k'}]] \\
&= 0 + 0 + 0 + [\sum_{k \in \mathcal{A}/\kappa} \hat{a}_k, \sum_{k' \in \mathcal{B}/\kappa'} \hat{b}_{k'}]] \\
&= [\sum_{k \in \mathcal{A}/\kappa} \hat{a}_k, \sum_{k' \in \mathcal{B}/\kappa'} \hat{b}_{k'}]]
\end{aligned}$$

In the last line, we see that the arguments of the commutator are nearly identical to the definitions of $\hat{A}$ and $\hat{B}$ but now the sets $\mathcal{A}, \mathcal{B}$ have been 'shrunk' by one element. By induction, we can repeat the same steps as before until all elements in either set have been exhausted, which results in the conclusion,

$$[\hat{A}, \hat{B}] = 0$$

C Proof of Plane-Wave Normalization Condition

An arbitrary optical field operator can be written as a linear combination of plane wave operators as,

$$\hat{a} = \sum_{j=s,p} \int_K d\mathbf{k} \phi_j(\mathbf{k}) \hat{a}_j(\mathbf{k})$$

where the normalization condition is,

$$\sum_{j=s,p} \int_K d\mathbf{k} |\phi_j(\mathbf{k})|^2 = 1$$

We prove this normalization condition is required by imposing the commutator requirement for any field operator $[\hat{a}, \hat{a}^\dagger] = 1$. The proof is direct.

$$\begin{aligned}
1 = [\hat{a}, \hat{a}^\dagger] &= \left[\sum_j \int d\mathbf{k} \phi_j(\mathbf{k}) \hat{a}_j(\mathbf{k}), \sum_{j'} \int d\mathbf{k}' \phi^*_{j'}(\mathbf{k}') \hat{a}^\dagger_{j'}(\mathbf{k}')\right] \\
&= \sum_{jj'} \iint d\mathbf{k} d\mathbf{k}' \phi_j(\mathbf{k}) \phi^*_{j'}(\mathbf{k}') [\hat{a}(\mathbf{k})_j, \hat{a}^\dagger_{j'}(\mathbf{k}')] \\
&= \sum_{jj'} \iint d\mathbf{k} d\mathbf{k}' \phi_j(\mathbf{k}) \phi^*_{j'}(\mathbf{k}') \delta_{jj'} \delta(\mathbf{k} - \mathbf{k}') \\
&= \sum_j \int d\mathbf{k} |\phi_j(\mathbf{k})|^2
\end{aligned}$$

D Proof of Bosonic Operator Commutators

Let $\hat{a}_1$ and $\hat{a}_2$ be annhilation operators for modes $\tilde{\phi}_{1j}(\mathbf{k})$ and $\tilde{\phi}_{2j}(\mathbf{k})$ represented in the plane wave basis for $j = s, p$. This means,

$$\hat{a}_1 = \sum_{j=s,p} \int_K d\mathbf{k} \tilde{\phi}_{1j}(\mathbf{k}) \hat{a}_j(\mathbf{k}) \tag{57}$$

$$\hat{a}_2 = \sum_{j=s,p} \int_K d\mathbf{k} \tilde{\phi}_{2j}(\mathbf{k}) \hat{a}_j(\mathbf{k}) \tag{58}$$

(59)

where the plane wave operators obey the commutation relation

$$[\hat{a}_j(\mathbf{k}), \hat{a}^\dagger_{j'}(\mathbf{k}')] = \delta_{jj'}\delta(\mathbf{k}-\mathbf{k}')$$

We now derive the commutator

$$\begin{aligned}
[\hat{a}_1, \hat{a}\dagger_2] &= \Bigg[\sum_{j=s,p}\int_K d\mathbf{k}\tilde{\phi}_{1j}(\mathbf{k})\hat{a}_j(\mathbf{k}), \sum_{j'=s,p}\int_{K'} d\mathbf{k}'\tilde{\phi}^*_{2j'}(\mathbf{k}')\hat{a}^\dagger_{j'}(\mathbf{k}')\Bigg] \\
&= \sum_{j=s,p}\sum_{j'=s,p}\int_K d\mathbf{k}\int K' d\mathbf{k}'\tilde{\phi}_{1j}(\mathbf{k})\tilde{\phi}^*_{2j'}(\mathbf{k}')[\hat{a}_j(\mathbf{k}), \hat{a}^\dagger_{j'}(\mathbf{k}')] \\
&= \sum_{j=s,p}\int_K d\mathbf{k}\tilde{\phi}_{1j}(\mathbf{k})\tilde{\phi}^*_{2j}(\mathbf{k}) \\
&= \sum_j \langle\tilde{\phi}_{2j}, \tilde{\phi}_{1j}\rangle
\end{aligned}$$

References

[1] P. Samphire, R. Loudon, and M. Babiker. “Quantum theory of radiation-pressure fluctuations on a mirror”. In: *Phys. Rev. A* 51 (4 Apr. 1995), pp. 2726–2737. DOI: 10.1103/PhysRevA.51.2726. URL: https://link.aps.org/doi/10.1103/PhysRevA.51.2726.